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Mathematical Reviews

Vol. 21, No. 4 April, 1960 Reviews 1930–2574

LOGIC AND FOUNDATIONS

See also 1947, 2224, 2559, 2560.

1930:

Holton, Gerald. A heuristic model for the growth process of modern physical science. Synthese **10**, 190–202.

It is proposed to construct a model for science which shall account for the nature of scientific discovery and concepts, the motivation of scientists and the cumulative growth of science. On discovery, Newton's laws of procedure are cited and it is shown that a strong irrational element also exists. In concepts, it is suggested that we distinguish between Public Science with its discipline of operational methods and clear concepts, and Private Science which is creative with freer license for imagination. In discussions of motivation one must invoke Private Science, such things being now quite neglected in the publications of Public Science. This dualism is not only in the life of the scientist but in the nature of science itself.

For these reasons a model is presented in which science acts like a growing organism, analogous to a biological species which may be regarded as an assembly of individuals or as a catalogued abstraction. There is a mechanism of continuity like heredity, identifiable with specific operational and quantitative concepts. Mutation is caused by boundless and unpredictable curiosity. Multiplicity of effort, much larger than the threshold, is necessary to secure sufficient production of lines that are ultimately useful and survive. Finally, there is a selection mechanism which enables a new and perhaps absurd idea to survive into a great innovation. Conflict is a fundamental necessity in the evolution of ideas.

D. Price (New Haven, Conn.)

1931:

★Lefebvre, Éloi. Structure et objet de l'analyse mathématique. Gauthier-Villars, Paris, 1958. x + 284 pp. $8.58.

Dieses sehr verdienstvolle Buch stellt sich in breiter Darstellung zur Aufgabe, die Struktur der mathematischen Analysis, deren Quellen und deren Ziele zu untersuchen.

Das erste Kapitel beschäftigt sich mit der Natur und der Rolle der konstitutiven (primitiven) Elemente der Analysis, d.h. deren "stofflichen" Material und den Operationen, die damit ausführbar sind. Im ersten Teil dieses Kapitels beschäftigt sich Verf. im wesentlichen mit der Mathematik im Hinblick auf die allgemeinen wissenschaftlichen Prinzipien. In der ersten Unterabteilung untersucht er die vormathematischen und voranalytischen Grundlagen der Analysis und die Rolle der Mathematik in der wissenschaftlichen Erkenntnis; in der zweiten, den Begriff des mathematischen Seins, die Ideen der Ursache und des Gesetzes und deren mathematischen Formen, und Begriffe der Quantität und der Grösse; und in der dritten, die allgemeinen logischen Forderungen, die Widerspruchslosigkeit, den Nutzen der eingeführten Begriffe und die Begriffen der Definition, des Postulates und der Hypothese.

Der zweite Teil des ersten Kapitels, der den mathematischen Operationen gewidmet ist, ist eingeteilt in vier Unterabteilungen. In der ersten Unterabteilung beschäftigt sich Verf. zuerst mit den Gegebenheiten der Arithmetik und Geometrie, welche Wissenschaften jede auf einer Form der Gleichheit gegründet sind. Es wird dann in der zweiten Unterabteilung übergegangen zur Analysis, als definiert durch einen neuen Begriff der Gleichheit, denjenigen, der zwei Grössen identifiziert, deren Auswertungen in einer unendlichen Folge von Operationen sich beliebig wenig unterscheiden lassen. Wir finden in dieser Unterabteilung die fundamentalen operativen Begriffe der Analysis: die Konvergenz und den Grenzwert, die Stetigkeit und die infinitesimale Ordnung, die Ableitung und die Gleichmässigkeit der Konvergenz. Entsprechend dem Charakter der Unendlichkeit der Analysis wird in der dritten Unterabteilung der Einbruch der Mengenlehre in die Analysis geschildert. In der vierten Unterabteilung werden die Definitionen eingeteilt in designative, limitative, operationelle und extensive. Viele Beispiele werden gegeben.

Das zweite Kapitel ist dem Studium der Beweise der Analysis gewidmet. Es werden besprochen: der analytische und synthetische Gesichtspunkt der deduktiven Methode; die zwei Formen des Beweisprozesses: die rein logische und die mathematisch-operative, die Rolle der gebrauchten Operationen und die Typen der Beweise. Dieser letzte Teil des Kapitels ist der am meisten entwickelte und umfasst nicht ganz ein Drittel des Buches. Es werden in ihm u.a. besprochen: Die Methode der Konstruktion eines Gegenbeispieles; Zerlegung des allgemeinen Falles in Sonderfälle; Unverträglichkeit von Eigenschaften; Erhaltung einer Eigenschaft im Laufe einer Transformation; Wiederholung einer Operation unter verschiedenen Bedingungen; stetige Variation eines Elementes; die Majoranten- und Minoranten-Methode; Einführung unbestimmter Elemente, uber die nachher verfügt wird.

Im dritten Kapitel schliesslich wird der Charakter der Resultate, nämlich der Sätze der Analysis, betrachtet.

B. Germansky (Berlin)

1932:

Kogalovskiĭ, S. R. Universal classes of models. Dokl. Akad. Nauk SSSR **124** (1959), 260–263. (Russian)

This paper establishes structural characteristics of universal classes of models which correspond to subcategories of the category of all models of a fixed type. The sense of the correspondence is that it allows, by

means of an inner mapping in the frame of the considered subcategory, the characterization of class of models which are universal in it. The result concerns classes of models with finite and countable sets of predicates, and often applies to models of a more general kind. The characterization is accomplished by defining for complete lattices a number of limit spaces (inf-spaces, sup-spaces) and using the properties of the topologies so defined.

E. J. Cogan (Bronxville, N.Y.)

1933:

Kurmitis, A. A. Independence of a system of axioms for the propositional calculus. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 21–25. (Russian. Latvian summary)

The author considers a list of eleven axioms for the propositional calculus closely allied to the one furnished in Kleene, *Introduction to metamathematics* [van Nostrand, New York, 1952; MR **14**, 525]. The axioms are listed so that the omission of the last leaves the intuitionistic calculus. It is shown that the full system is redundant as formulated, and normal tables of values are provided to demonstrate the independence of each of the axioms save one. The only axiom whose independence value has more than four values is A2 (Kleene's A1b), and this is an eight-valued table. The implications of this are discussed at the end of the paper. *E. J. Cogan* (Bronxville, N.Y.)

1934:

Lévy, Azriel. Comparison of subtheories. Proc. Amer. Math. Soc. **9** (1958), 942–945.

If Q and T are first-order theories, then Q is an "essentially infinite" extension of T if Q is a consistent extension of T and no consistent extension of Q without new symbols is a finite extension of T. The author shows that the existence of a special kind of truth definition implies that, for two related systems, one is an essentially infinite extension of the other. A consequence of this theorem is that if S_1 is Zermelo-Fraenkel set theory and S_2 is Zermelo set theory (that is, S_1 with the Aussonderungsaxiom instead of the Replacement Axiom), then, if S_1 is consistent, the number theory of S_1 is an essentially infinite extension of the number theory of S_2. A similar result holds when S_2 is Zermelo-Fraenkel set theory, and S_1 is S_2 plus an axiom asserting the existence of inaccessible ordinals. *E. Mendelson* (New York, N.Y.)

1935:

Yablonskiĭ, S. V. On the impossibility of eliminating trial of all functions from P_2 in solving some problems of circuit theory. Dokl. Akad. Nauk SSSR **124** (1959), 44–47. (Russian)

Let Q be a class of functions from algebraic logic (P_2), let $L(f, n)$ be the least number of contacts in a circuit realization of $f(x_1, \cdots, x_n)$, and let $L_Q(n) = \max L(f, n)$ for f in the class Q. The problem whose investigation leads to the results of the paper is to learn more accurately the size of $L_{P_2}(n)$ for each natural number n. To do this, the author attempts to construct, allowing a certain class of algorithms, a sequence M^0 of functions

$$\{f_1^0(x_1), f_2^0(x_1, x_2), f_3^0(x_1, x_2, x_3), \cdots\}$$

such that there is a subsequence $\{n_k\}$ such that

$$L(f_{n_k}^0, n_k)/L_{P_2}(n_k) \to 1.$$

The difficulties encountered in the construction of M^0 lead to the results mentioned in the title.

E. J. Cogan (Bronxville, N.Y.)

1936:

Yablonskiĭ, S. V. Some properties of enumerable closed classes from $P_{\aleph_0}$. Dokl. Akad. Nauk SSSR **124** (1959), 990–993. (Russian)

The class $P_{\aleph_0}$ contains functions from natural number n-tuples to natural numbers. Let $\mathscr{P}$ be a subclass closed under function composition (superposition). A set of functions in $\mathscr{P}$ is said to be complete if its compositions span $\mathscr{P}$ (in the vector space sense). Such a set is a basis if it is complete, but no proper subset of it is complete. A class $\mathscr{P}$ is called a limited logic if it is countable and contains homomorphic images of all k-valued logics for $k = 2, 3, \cdots$. Some results of this paper are the following: (1) There exists a limited logic having no basis; (2) there exists a limited logic which has a basis, but such that it contains complete sets of functions of which no subset is a basis; (3) each countable closed class of functions of $P_{\aleph_0}$ can be imbedded in a limited logic spanned by a single function. *E. J. Cogan* (Bronxville, N.Y.)

1937:

Bottenbruch, H. Übersetzung von algorithmischen Formelsprachen in die Programmsprachen von Rechenmaschinen. Z. Math. Logik Grundlagen Math. **4** (1958), 180–221.

A study, in suitable Turing machine terminology, of mechanisms of transition from the (mathematician's) formulation language (for algorithmic problems) into the (computing machine's) execution language, and the extension of the latter by what may be regarded as recursive subprogramming. The discussion is thoroughly illustrated.

R. M. Baer (Berkeley, Calif.)

SET THEORY

1938:

Levy, Azriel. A note on definitions of finiteness. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 83–84.

Two possible definitions of finiteness are considered: (I*) For any subset B of A at least one of B and $A - B$ is finite (in the ordinary sense of being equivalent to the set of positive integers less than some integer); (II) Each non-void monotonic set of subsets of A has a maximal member. Ordinary finiteness implies (I*), and (I*) implies (II). Given the axiom of choice, they are all equivalent. The author shows that (I*) is equivalent to (I_1*): Each non-void set of subsets of A has at least one maximal or one minimal member. He also proves that (II) is equivalent to several other possible definitions, one of which is (II_3): Each monotonic set of subsets of A is finite.

E. Mendelson (New York, N.Y.)

1939:

Eyraud, Henri. Les ordinaux de la troisième classe et le problème du continu. Cahiers Rhodan. **7** (1955/56), 8 pp.

"La première tentative d'utiliser la représentation des ordinaux de la troisième classe pour démontrer le Théorème du Continu est déjà ancienne (1944). Elle est reprise ici en utilisant la notion récente de fonctionelle caractéristique (1955)." (Résumé de l'auteur.)

F. Bagemihl (South Bend, Ind.)

1940:

Eyraud, H. Ensembles agrégatifs ou intersectifs. I. Cahiers Rhodan. **8** (1957/58), 22 pp.

Cf. Eyraud, *Leçons sur la théorie des ensembles, les nombres transfinis et le problème du continu*, 2d ed., Institut de Mathématiques, Lyon, 1949; Ann. Univ. Lyon. Sect. A. (3) **14** (1951), 5–28 [MR **12**, 166; **14**, 255].

F. Bagemihl (South Bend, Ind.)

COMBINATORIAL ANALYSIS

See also 2339, 2340.

1941:

Hyltén-Cavallius, C. On a combinatorical problem. Colloq. Math. **6** (1958), 59–65.

The author improves on previous estimates for the maximum number of ones possible in an $m \times n$ incidence $(0, 1)$ matrix with no $2 \times k$ minor consisting of all ones. He shows that this maximum is less than or equal to $\frac{1}{2}n + \{(k-1)nm(m-1) + \frac{1}{4}n^2\}^{1/2}$ and from this derives various asymptotic results.

K. Goldberg (Washington, D.C.)

1942:

Ford, Lester R., Jr.; and Johnson, Selmer M. A tournament problem. Amer. Math. Monthly **66** (1959), 387–389.

This paper deals with the smallest number of paired comparisons necessary to rank n objects according to some transitive characteristic. Let $U(n)$ be an upper bound to this number. A recurrence formula and an asymptotic formula for $U(n)$ are obtained. The formulas improve a result of H. Steinhaus [*Mathematical snapshots*, Oxford Univ. Press, New York, 1950; MR **12**, 44; pp. 37–40].

N. S. Mendelsohn (Winnipeg)

1943:

Hanani, Haim. On the number of monotonic subsequences. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 11–13.

It is proved that if $g(m)$ represents the minimum number of (not necessarily strictly) monotonic subsequences into which every sequence of m real numbers may be decomposed, then $g(m) \leqq n$ provided $m \leqq \frac{1}{2}n(n+3)$. The proof proceeds by an induction on n. If $m = \frac{1}{2}n(n+3)+1$ then there exists a sequence of m terms which is not decomposable into n or less monotonic subsequences. The theorem is used to give a simple proof of a theorem of P. Erdös and G. Szekeres [Compositio Math. **2** (1935), 463–470] which gives a lower bound on the maximum number of terms in a monotonic subsequence of a sequence of real numbers.

J. K. Goldhaber (St. Louis, Mo.)

1944:

Hajnal, András; und Surányi, János. Über die Auflösung von Graphen in vollständige Teilgraphen. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 113–121.

Let G be a finite or infinite graph. Assume that the maximum number of independent points is $k < \infty$. Also assume that in every non-self-intersecting polygon of at least four edges at least one diagonal is present. Then there exist at most k complete graphs such that every vertex is incident to at least one of these. This generalizes an unpublished theorem of Gallai. If it is only assumed that the number of independent points is finite and the condition on polygons is retained, then G is the union of denumerably many complete graphs. It is stated without proof that if the maximum number of independent points is denumerable and the diagonal condition is satisfied, then, assuming the continuum hypothesis, the graph is the union of $\aleph_1$ complete graphs. It is an open question whether it is the union of $\aleph_0$ complete graphs and this question is shown to be related to Souslin's hypothesis.

L. Moser (Edmonton, Alta.)

ORDER, LATTICES

See also 1944, 1946, 1947.

1945:

Kolibiar, M. Bemerkung über die Ketten in teilweise geordneten Mengen. Acta Fac. Nat. Univ. Comenian. **3**, 17–22 (1958). (Slovak and Russian summaries)

O. Ore [Bull. Amer. Math. Soc. **49** (1943), 558–566; MR **5**, 88] developed for partially ordered sets an analog of the Jordan theorem about equality of lengths of chains. This was extended by S. MacLane [ibid. **49** (1943), 567–568; MR **5**, 88] and M. Benado [Rev. Math. Pures Appl. **1** (1956), 5–12; MR **18**, 275]. The present paper partially removes requirements of finite length.

Let P be a partially ordered set containing two chains $u \leqq a \leqq v$, $u \leqq b \leqq v$. This pair of chains is called a quadrilateral $(u, v; a, b)$; it is called simple if there are no p, q, $r \in P$ with $a \leqq p < v$, $b \leqq q < v$, $u < r \leqq p$, $r \leqq q$, or dually. A pair of maximal chains A, B from u to v $(u < v)$ is called a cycle; it is called irreducible if for every $a \in A$, $b \in B$ with $a, b \neq u, v$, the quadrilateral $(u, v; a, b)$ is simple. Consider the proposition (A): If there is a finite maximal chain in P from u to v then all maximal chains from u to v are finite and have the same length. Theorem: (A) holds in P if and only if (I) or (II) holds: (I) If a chain of an irreducible cycle is finite, so is the other chain of this cycle and they have the same length; (II) If for a simple quadrilateral $(u, v; a, b)$, $a, b \neq u, v$, there is a finite maximal chain from u to v via a, then there is a cycle, with a in one of its chains and b in the other, whose component chains are finite and of equal length. Corollary: If (I) or (II) holds and there is a finite maximal chain A from u to v, then every chain from u to v is finite and obtainable from A by a succession of "simple deformations", i.e., replacements of a part of a chain by another forming an irreducible cycle with it.

P. M. Whitman (Silver Spring, Md.)

GENERAL MATHEMATICAL SYSTEMS

1946:

Ponasse, Daniel. Structure prébooléienne. Ensemble prébooléien universel associé à un ensemble quelconque. C. R. Acad. Sci. Paris **248** (1959), 899–902.

L'auteur définit un ensemble prébooléien comme un ensemble F muni: (1) d'une opération $x \cdot y$; (2) d'une application biunivoque $x \to -x$ de F dans F; et (3) d'une famille non vide $(C_t)_I$ (dite la famille déductive de F) de parties de F telles que pour tout t: (a) $x \in C_t \Leftrightarrow -x \notin C_t$; (b) $x \cdot y \in C_t \Leftrightarrow x \in C_t$ et $y \in C_t$. Si l'on pose $T = \bigcap_I C_t$, la relation R exprimée par "$-(x \cdot -y) \in T$ et $-(y \cdot -x) \in T$" est une relation d'équivalence dans F, compatible avec les opérations (1) et (2). F/R muni de la structure quotient est un anneau booléien (dit engendré par F) ayant un élément unité, et tout anneau booléien B ayant un élément unité peut être regardé comme un ensemble prébooléien en prenant comme l'opération (1) la multiplication dans B, comme $-x$ le complément de x et comme famille déductive la famille de tous ses ultrafiltres. Étant donnés deux ensembles prébooléiens $F(T, R)$ et $F'(T', R')$, on appelle préhomomorphisme de F dans F' tout application f telle que: 1. f conserve les opérations (1) et (2) modulo R' dans F'; 2. $f(T) \subset T'$. Un préhomomorphisme f induit un homomorphisme d'anneaux de F/R dans F'/R'.

Étant donné un ensemble E quelconque, l'auteur construit un ensemble prébooléien $\dot{E}$ et une application h de E dans $\dot{E}$ avec la propriété suivante: toute application f de E dans un ensemble prébooléien quelconque F est de la forme $f = g \circ h$ où g est un préhomomorphisme de $\dot{E}$ dans F. *M. Kolibiar* (Bratislava)

1947:

Ponasse, Daniel. Anneau booléien universel associé à un ensemble quelconque. Applications au calcul propositionnel. C. R. Acad. Sci. Paris **248** (1959), 1093–1096.

Ce travail constitue la suite d'un précédent travail [cf. analyse précédente; mêmes notations que ci-dessus]. L'anneau booléien $\dot{E}/R$ engendré par $\dot{E}$, noté $\langle E \rangle$, est (à isomorphisme près) un anneau booléien libre avec le système E de générateurs. On donne des applications de ce qui précède au calcul propositionnel. On désigne par $\mathscr{A}$ l'ensemble de formules se réduisant à une variable propositionnelle, par $\mathscr{F}$ l'ensemble de toutes les formules et par $\wedge$ et $-$ la conjonction et la négation. $\mathscr{F}$ est un ensemble prébooléien pour les opérations $\wedge$ et $-$ et pour la famille déductive constituée par tous les systèmes déductifs compatibles et complets. L'anneau booléien $\langle \mathscr{F} \rangle$ peut être regardé comme un ensemble prébooléien en prenant comme l'opération (1) la multiplication dans $\langle \mathscr{F} \rangle$, comme $-x$ le complément de x et comme famille déductive—une famille spéciellement construite. L'anneau booléien, noté $\{\mathscr{F}\}$, engendré par cet ensemble prébooléien, est isomorph à $\langle \mathscr{A} \rangle$. Il existe un préhomomorphisme H' de $\mathscr{F}$ dans $\langle \mathscr{A} \rangle$ qui induit un isomorphisme des anneaux booléiens $\mathscr{F}/R$ et $\langle \mathscr{A} \rangle$. Soi G une formule. L'auteur note que la connaissance de $H'(G)$ permet de faire immédiatement l'analyse fonctionnelle des valeurs de vérité de G, de mettre G sous la forme normale conjonctive, ou de résoudre si G est démontrable, et que l'étude de $\{\mathscr{F}\}$ et de l'isomorphisme de $\langle \mathscr{A} \rangle$ dans $\{\mathscr{F}\}$ permet en particulier de calculer $H'(G)$. *M. Kolibiar* (Bratislava)

THEORY OF NUMBERS

1948:

da Costa, Newton Carneiro Affonso. Une généralisation du théorème de Bouniakowsky. Soc. Parana. Mat. Anuário **3** (1956), 12–16.

The theorem of Bouniakowsky is: $(k-1)!(p-k)! \equiv (-1)^k \pmod p$, $k \leq p$, p prime. The author gives an extension of this theorem. Let $a_1, a_2, \cdots, a_{\varphi(n)}$ ($\varphi(n)$ is Euler's totient function), $a_q < a_{q+1}$, be the numbers $< n$ and prime to n; then $a_1 a_2 \cdots a_{\varphi(n)} \equiv -1 \pmod n$ if n is a number "of the first class"—that is to say, if n is 2, 4, p^k or $2p^k$ (p a prime >2)—and $\equiv +1 \pmod n$ if n is any other number ("of the second class"). This is a known theorem. The extension is: Let $k \leq \varphi(n)+1$; then $a_1 a_2 \cdots a_{k-1} \cdot a_1 a_2 \cdots a_{\varphi(n)-k+1} \equiv (-1)^k \pmod n$ if n is a number of the first class, and $\equiv (-1)^{k+1} \pmod n$ if n is a number of the second class. A proof is given by induction on k, using the relation $a_{\varphi(n)-q+1} = n - a_q$. Nearly 15 consequences are given; e.g.,

$$\left(\frac{p+1}{2}\right)^2 \cdot \left(\frac{p+3}{2}\right)^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \pmod p,$$

p prime. *N. G. W. H. Beeger* (Amsterdam)

1949:

da Costa, Newton Carneiro Affonso. Some elementary theorems on divisibility. Soc. Parana. Mat. Anuário **3** (1956), 60–63. (Portuguese)

$\dot{p}$ denotes a multiple of p, and $n!!$ seems to signify $1 \cdot 3 \cdot 5 \cdots n$ if n is odd and $2 \cdot 4 \cdot 6 \cdots n$ if n is even. Eight theorems are proved. Here are specimens:

$$(2p-1)!/p! + 1 \equiv 0 \pmod p;$$

$$[(2p-1)!!/p!!]^2 + (-1)^{(p-1)/2} \equiv 0 \pmod p;$$

$$1!2! \cdots (p-1)! \equiv \left(\frac{p-1}{2}\right)! \, (-1)^{(p-1)/8} \pmod p.$$

N. G. W. H. Beeger (Amsterdam)

1950:

Shanks, Daniel; and Wrench, J. W., Jr. Khintchine's constant. Amer. Math. Monthly **66** (1959), 276–279.

Khintchine's constant

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right)^{\log n/\log 2} = 2.68545\cdots,$$

which is of importance in the theory of continued fractions, is evaluated (together with its logarithm) to some 65 decimal places by making the calculations depend on the values $\zeta(2k)$, $1 \leq k \leq 68$. *M. Newman* (Washington, D.C.)

1951:

Copley, G. N. Recurrence relations for solutions of Pell's equation. Amer. Math. Monthly **66** (1959), 288–290.

Making use of the usual expression for solutions of the Pell equation in terms of the fundamental one, the author quickly deduces some recurrence relations. Relationships with the circular and hyperbolic functions are exploited.

B. W. Jones (Boulder, Colo.)

1952:

Stolt, Bengt. Die Anzahl von Lösungen gewisser diophantischer Gleichungen. Arch. Math. **8** (1957), 393–400.

The author discusses the equation (1) $x^3+x+\frac{1}{4}(D+1)=y^q$, where q is an odd prime and D a positive integer, $D\equiv 3 \pmod 4$ and square-free. Let h denote the number of ideal classes in the field $K(\sqrt{-D})$ and $D_1=\frac{1}{4}(D+1)$. For given values of D and q the following theorem is proved: (1) has at most one solution in positive integers x, y if $(h, q)=1$, and either D_1 is even or D_1 is odd and at the same time $q\equiv 5 \pmod 6$. (1) has at most three solutions if D_1 is odd and $q\equiv 1 \pmod 6$. This generalizes a result due to B. Persson [Ark. Mat. **1** (1949), 45–57; MR **11**, 328]. In the proof use is made of an idea of Persson.

W. Ljunggren (Oslo)

1953:

Mikolás, Miklós. Über die Charakterisierung der Hurwitzschen Zetafunktion mittels Funktionalgleichungen. Acta Sci. Math. Szeged **19** (1958), 247–250.

Let $f(s)$ and $g(s)$ $(s=\sigma+it)$ both be representable as Dirichlet series which converge absolutely for $\sigma>1$, let $f(s)$ satisfy a certain condition concerning analytic continuation over the whole plane, and let

$$g(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}=f(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{(s-1)/2}.$$

Then $f(s)=g(s)=c\zeta(s)$ (c a constant) [H. Hamburger, e.g. Math. Z. **10** (1921), 240–254]. Here a similar problem is dealt with, but from a different point of view, for the Hurwitz zetafunction

$$\zeta(s,u)=\sum_{n>-u}^{\infty}(u+n)^{-s} \quad (\sigma>1,\ u \text{ real}).$$

Clearly $\zeta(s,u+1)=\zeta(s,u)$. The author proves: if $\Phi(s,u)$ is an entire function of s and a periodic and differentiable function of the real variable u, and belongs to $\mathrm{Lip}_K 1$ with respect to s $(s>1)$ for any real u; if furthermore

$$(\partial/\partial u)\Phi(s+1,u)=\Phi(s,u),$$

$$\int_0^1\Phi(s_1,v)\Phi(s_2,u-v)dv=\Phi(s_1+s_2,u) \quad (1<s,s_1,s_2<2),$$

and $c_1(s)\not\equiv 0$, $|\arg c_n(s)|<\pi$, where $1<s<2$, $n=1, 2, \cdots$, $c_n(s)=\int_0^1\Phi(s,v)e^{-2\pi inv}dv$; then $\Phi(s,u)\Gamma(s)=\zeta(1-s,u)$. Finally the analogy of this result to a known property of the Bernoulli polynomials is indicated.

H. Kober (Birmingham)

1954:

Kuga, Michio. Topological analysis and its applications in weakly symmetric Riemannian spaces. (Introduction to the work of A. Selberg.) Sûgaku **9** (1957/58), 166–185. (Japanese)

This is the reproduction in Japanese, with several remarks, of a work of A. Selberg [J. Indian Math. Soc. (N.S.) **20** (1956), 47–87; MR **19**, 531]. *K. Yano* (Tokyo)

1955:

Świerczkowski, S. On the intersection of a linear set with the translation of its complement. Colloq. Math. **5** (1958), 185–197.

Let N be a positive integer, let A and B be complementary subsets of $\{1, 2, \cdots, N\}$; $|A|$ = number of elements of A, etc. Let M_n denote the number of solutions of $n=a-b$, $a\in A$, $b\in B$. Then

$$(1) \qquad \max_n M_n \geq \frac{N}{5}(2-[4-10|A|\,|B|/N^2]^{1/2}).$$

Let I be an interval. Let X and Y be complementary Lebesgue measurable subsets of I; mX = measure of X, etc. Let m_t denote the measure of $\{x|x\in X,\ x+t\in Y\}$. As a corollary of (1) the author also proves: There is a t such that

$$m_t \geq \frac{mI}{5}(2-[4-10\cdot mX\cdot mY/(mI)^2]^{1/2}).$$

P. Scherk (Boulder, Colo.)

1956:

Browkin, J. Sur les décompositions des nombres naturels en sommes de nombres premiers. Colloq. Math. **5** (1958), 205–207.

Let $P(n)$ denote the number of representations of n as a sum of primes, repetitions being allowed but order being regarded as irrelevant. A study of generating functions associated with $P(n)$ and with other partition functions recently enabled P. T. Bateman and P. Erdös [Publ. Math. Debrecen **4** (1956), 198–200; Mathematika **3** (1956), 1–14; MR **18**, 15, 195] to show that $P(n+1)\geq P(n)$ for $n\geq 1$ and that $P(n+1)-P(n)\to\infty$ as $n\to\infty$. In the paper under review, the author gives a new proof of these results and shows, moreover, that $P(n+1)>P(n)$ for $n\geq 8$. The argument is entirely elementary and surprisingly simple.

L. Mirsky (Sheffield)

1957:

Vinogradov, A. I. Estimates for binary problems. Vestnik Leningrad. Univ. **14** (1959), no. 7, 26–31. (Russian. English summary)

This paper is concerned with approximations to Goldbach's conjecture. Let N be a positive integer and α a positive number. The author states that, if $\alpha<\theta$ (where θ is a certain constant approximately equal to 5/22), then N can be represented as the sum of two integers all of whose prime divisors exceed N^α. (This is said to follow Selberg's sieve method.) He then indicates, in bare outline only, how θ could (in principle) be replaced by a larger constant. The argument is involved, and relies, among other things, on Selberg's method and on complicated estimates for triple contour integrals. The reviewer was not successful in following the details.

L. Mirsky (Sheffield)

1958:

Wang, Yuan. On sieve methods and some of their applications. I. Acta Math. Sinica **8** (1958), 413–429. (Chinese. English summary)

The author gives fully detailed proofs of the following three theorems. (1) Every sufficiently large even number can be written as a sum $a+b$ where $a\geq 2$ has at most two and $b\geq 2$ at most three prime factors. (2) If k is any even number, there exist infinitely many integers n such that $n(n+k)$ has at most five prime factors, while neither n nor $n+k$ has more than three prime factors. (3) Every sufficiently large odd number can be written as a sum $2P+Q$ where P, Q, and PQ have at most three, three, and five prime factors, respectively. *K. Mahler* (Manchester)

1959:

Yüh, Ming-i. A divisor problem. Acta Math. Sinica **8** (1958), 496–506. (Chinese. English summary)

The paper gives details of the proof that the remainder term in the asymptotic formula for $\sum_{n\leq x} d_3(n)$ is $O(x^{14/29+\varepsilon})$; the result was announced previously in Sci. Record (N.S.) **2** (1958), 326–328 [MR **21** #35].

F. V. Atkinson (Canberra)

1960:

Karamata, J. Sur les inversions asymptotiques des produits de convolutions. Glas Srpske Akad. Nauka **228** Od. Prirod.-Mat. Nauka (N.S.) **13** (1957), 23–59. (Serbo-Croatian. French summary)

Define

$$F(x) = \sum_{n\leq x} f(n), \quad G(x) = \sum_{n\leq x} F(x/n) = \sum_{n\leq x} f(n)[x/n],$$

where $[w]$ denotes the greatest integer contained in w. What are the conditions on $p(x)$ so that from (1) $G(x) = Ax \log x + p(x)$ we can deduce that (2) $F(x) = Ax + o(x)$? The author discusses known results in this direction, and obtains the following: Theorem 5: If

$$p(x) = Bx + Cx/\log x + o(x/\log x)$$

and if $f(n) \geq -M$ $(n = 1, 2, 3, \cdots)$, then (1)→(2). The idea of the proof is that of A. Selberg in his elementary proof of the Prime Number Theorem [Ann. of Math. (2) **50** (1949), 305–313; MR **10**, 595]. See also the succeeding review.

S. Chowla (Boulder, Colo.)

1961:

Karamata, J. Sur les inversions asymptotiques de certains produits de convolution. Bull. Acad. Serbe Sci. (N.S.) **20** Cl. Sci. Math.-Nat. Sci. Math. **3** (1957), 11–32.

The author proves (we use the notation of the preceding review) the sharper result: If $p(x) = Bx + o(x)$, then $\int_1^x t^{-1}F(t)dt = Ax + o(x)$; and if, in addition, $f(n) \geq -M$, then equation (2) (of the preceding review) holds.

S. Chowla (Boulder, Colo.)

1962:

Mineev, M. P. A Diophantine equation involving an exponential function and its application to the study of an ergodic sum. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 585–598. (Russian)

Let $g \geq 2$, $m_1, \cdots, m_k$, $n_1, \cdots, n_k$ be fixed positive integers. The author proves that there are $cp^k + O(p^{k-1})$ integral solutions $(x_1, \cdots, x_k, y_1, \cdots, y_k)$ of the conditions

$$m_1 g^{x_1} + \cdots + m_k g^{x_k} = n_1 g^{y_1} + \cdots + n_k g^{y_k},$$
$$0 \leq x_1, \cdots, x_k, y_1, \cdots, y_k \leq p-1,$$

as $p \to \infty$; here $c \geq 0$ does not depend on p. For $m_1 = \cdots = m_k = n_1 = \cdots = n_k = 1$ this was found by A. G. Postnikov [Festschrift anlässlich des 250. Geburtstages Leonhard Eulers, Berlin, 1957]. The following application is made. Let $f(t)$ be a real function of period 1 satisfying

$$\int_0^1 f(t)dt = 0, \quad \int_0^1 f(t)e^{2\pi i n t}dt = O(|n|^{-\beta}) \quad (|n| \to \infty),$$

where $\beta > 1/2$, such that $\sigma > 0$, where

$$\sigma^2 = \int_0^1 f(t)^2 dt + 2\sum_{k=1}^{\infty} \int_0^1 f(t)f(g^k t)dt.$$

Then

$$\lim_{p\to\infty} p^{-1} \int_0^1 \left(\sum_{k=0}^{p-1} f(g^k t)\right)^2 dt = \sigma^2,$$

and the author proves that

$$\lim_{p\to\infty} \operatorname{meas}_{0\leq\alpha<1} E\left\{\sum_{x=0}^{p-1} f(g^x\alpha) < \lambda\sqrt{p}\right\} = \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\lambda} e^{-z^2/2\sigma^2} dz,$$

by first considering finite trigonometric sums.

K. Mahler (Manchester)

1963:

Gordeev, N. V. On the mean length of intervals consisting of numbers of the same quadratic character with respect to a prime modulus. Bilsk. Gos. Ped. Inst. Uč. Zap. **1** (1957), 111–120. (Russian)

The author shows that for an odd prime p the mean number of consecutive integers in the interval $(1, p-1)$ having the same quadratic character is equal to 3. To prove this he obtains an expression for P, the number of isolated quadratic residues and non-residues, and Q, the number of sets of consecutive integers having the same quadratic character. The average number of consecutive integers in each set of Q is then given by $T = (p-1-P)/Q$. T is of the form $(3p+a)/(p+b)$, where a and b are determined constants, dependent upon whether $p \equiv 1$, 3, 5 or 7 (mod 8). Thus $T \to 3$ as $p \to \infty$. The expressions for P and Q are obtained by evaluating certain sums of Legendre symbols.

W. H. Simons (Vancouver, B.C.)

1964:

Delcourte, M. Sur les résidus quadratiques distincts d'un nombre. Mathesis **67** (1958), 343–347.

Let $R(m)$ denote the number of distinct quadratic residues modulo m between 0 and $m-1$. The author shows that R is a multiplicative function, and can be specified by the formulae

$$R(p^n) = \frac{2p^{n+1} - (-1)^n(p-1)}{4(p+1)} + \frac{3}{4},$$

$$R(2^n) = \frac{2^n - (-1)^n}{6} + \frac{3}{2}.$$

His proof is a more detailed version of the one given by the reviewer [Amer. J. Math. **68** (1946), 560–580; MR **8**, 255; see p. 563].

R. C. Buck (Los Angeles, Calif.)

1965:

Turán, P. Über die Potenzsummen komplexer Zahlen. Arch. Math. **9** (1958), 59–64.

The author has considered in his book [*Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688] and in other papers, the quantity

$$M_n = \min \max_{1\leq\nu\leq n} |s_\nu| = \min \max_{1\leq\nu\leq n} |z_1^\nu + \cdots + z_n^\nu|,$$

where the minimum is extended over all the sets of complex numbers $z_1, \cdots, z_n$ with $z_n = 1$. It has been conjectured that $M_n > c > 0$, but all that is known is that $M_n > (1-\varepsilon)\log\log n/\log n$. In the present paper the related quantity

$$m_n = \min_{z_n=1} \sum_{j=1}^{n} \frac{|s_j|^2}{j}$$

is considered. It is shown that $m_n \leqq e^4$. Moreover, if $\xi_1, \cdots, \xi_{n-1}, 1$ is a set of nonzero values of $z_1, \cdots, z_n$ for which the above sum is minimal, it is shown that the polynomial $\psi(z)=(1-\bar{\xi}_1 z)\cdots(1-\bar{\xi}_{n-1}z)(1-z)$ satisfies the differential-functional equation

$$\lambda\psi'(z)-\bar{\lambda}z^{2n-1}\bar{\psi}'(z^{-1}) = z^n(1-z)^{-1}\psi(z)\bar{\psi}(z^{-1}),$$

with $\psi(1)=0$ and suitable constant $\lambda=\lambda(n)$.

W. J. LeVeque (Ann Arbor, Mich.)

1966:

Macbeath, A. M.; and Rogers, C. A. Siegel's mean value theorem in the geometry of numbers. Proc. Cambridge Philos. Soc. **54** (1958), 139–151.

Let T_n be the set of real $n\times n$ matrices with unit determinant, G_n the unimodular group in T_n, F_n a fundamental domain for T_n/G_n and $\tilde{\mu}$ the Siegel measure in T_n. Further let LI be the set of $n\times l$ matrices with integer elements and linearly independent columns, $1\leqq l\leqq n-1$, and μ the Lebesgue measure in X, the set of real $n\times l$ matrices. Siegel [Ann. of Math. (2) **46** (1945), 340–347; MR **6**, 257] stated without proof that for any Lebesgue integrable function defined on X,

$$\int_{F_n}\sum_{q\in \mathrm{LI}} f(tq)\,d\tilde{\mu}(t) = \tilde{\mu}(F_n)\int_X f(x)\,d\mu(x). \tag{1}$$

The authors give a proof of this result. In case $l=1$, (1) becomes the mean value theorem for lattices proved by Siegel [loc. cit.]. The authors state that their main object is a simplified version of this proof. Under the inductive assumption that $\tilde{\mu}(F_m)$ is finite for $m<n$ they arrive at a mean value formula for primitive matrices. Applying elementary geometry and Lebesgue's convergence theorem to this formula they deduce the identity

$$\tilde{\mu}(F_n) = \left(\frac{n-1}{n}\right)\zeta(n)\tilde{\mu}(F_{n-1}).$$

The formula for primitive matrices is thus induced and the proof of (1) is completed in a simple way. With this approach and a separate proof of the existence of a fundamental domain the authors point out that Minkowski's reduction theory is unnecessary to the argument. For another approach see Weil [Summa Brasil. Math. **1** (1946), 21–39; MR **7**, 411]. *A. C. Woods* (New Orleans, La.)

FIELDS

See also 2157.

1967:

Leopoldt, Heinrich-Wolfgang. Über Klassenzahlprimteiler reeller abelscher Zahlkörper als Primteiler verallgemeinerter Bernoullischer Zahlen. Abh. Math. Sem. Univ. Hamburg **23** (1959), 36–47.

The principal result of the paper is contained in the following.

Theorem 1. Let K be a real abelian algebraic number field of degree n and let l be a prime, $l\nmid 2n$, that divides the class-number of K. Then

$$B_K(l) \equiv 0 \pmod{l}. \tag{*}$$

The generalized Bernoulli number $B_K(l)$ is defined as follows. Let χ denote a primitive character (mod m) and put

$$\sum_{s=1}^{m}\chi(s)\frac{te^{st}}{e^{st}-1} = \sum_{n=0}^{\infty}B_{\chi^n}\frac{t^n}{n!}$$

[Leopoldt, same Abh. **22** (1958), 131–140; MR **19**, 1161]. Now if $R<Z\leqq K$, where R denotes the rational field, and χ is a generating character of Z, put $\chi=\psi\chi_0$, where the conductor of ψ is a power of l, while the conductor of χ_0 is prime to l; let e denote the order of ψ so that $e\mid l-1$. Then put

$$B^Z(l) = \prod_{\substack{r=1\\(r,s)=1}}^{s}\prod_{\chi_0}{}' B_{\chi_0}{}^{r(l-1)/s},$$

the inner product extending over all χ_0 with conductor prime to l such that $\chi=\psi\chi_0$. Note that $B^Z(l)$ is integral (mod l) and that $B^Z(l)\neq 0$. Finally put

$$B_K(l) = \prod_{R<Z\leqq K} B^Z(l).$$

In particular when K is the maximal real subfield of the field $R(e^{2\pi i/l})$, then (*) reduces to the well-known criterion of Kummer.

It is pointed out that theorem 1 can also be formulated in the following way. Let K be a real abelian field of degree $n>2$ and let $\zeta_K(s)$ denote the Dedekind zeta-function of K. Then if the prime l, $l\nmid 2n$, is a divisor of the class-number of K, it follows that the rational number

$$\pi^{-n(l-1)}\zeta_K(l-1)$$

is integral (mod l).

As an application of theorem 1, the author proves the following.

Theorem 2. Let $K=R(p^{1/2})$, where p is a positive prime $\equiv 1 \pmod 4$ and let $\chi(r)=(r/p)$, the Legendre symbol. Then if the rational numbers $B_{\chi}{}^{l-1}$ are all prime to l, for all odd primes $l<\frac{1}{2}p^{1/2}\log p$, it follows that the class-number of K is equal to 1.

L. Carlitz (Durham, N.C.)

1968:

Fujisaki, Genjiro. On an example of an unramified Galois extension. Sûgaku **9** (1957/58), 97–99. (Japanese)

The author gives an example of a Galois extension K/k such that (i) the class number of k is 1 and (ii) every finite prime divisor of k is unramified in K/k. Let Q be the rational field, θ be a root of the equation $X^5-X-1=0$. Then $K=Q(\theta)$ is a Galois extension of Q with the Galois group $\mathfrak{S}_5$ (symmetric group). The discriminant of this equation is $2869=19\times 151$. Let $k=Q(\sqrt{2869})$. Then the author proves that K/k is a required example with the Galois group $\mathfrak{A}_5$ (alternating group).

Y. Kawada (Tokyo)

1969:

Nobusawa, Nobuo. On a system of valuations of algebraic function fields with several variables. Osaka Math. J. **10** (1958), 205–212.

Suppose that $F=k(x_1, \cdots, x_n)$ is a rational function field of n variables over a field of characteristic 0. The author terms a set of n elements t_i of F a set of prime elements for a valuation w of rank n on F if $w(t_i)=(0, \cdots, 1, \cdots, 0)$. Next a subset S in the set of all rank n valuations of F/k is defined by the requirements that there exist prime elements t_i for $w\in S$ such that $t_i\in k(x_i)$ and t_i either equal x_i-a_i ($a_i\in k$) or $1/x_i$, except for one t_j. Using simple

inductive arguments the author proves that the elements $y \in F$ can be distinguished by the valuations of S, i.e., for y there exist w, $w' \in S$ with $w(y) > (0, \cdots, 0)$, $w'(y) < (0, \cdots, 0)$; thus $y \in k$ if and only if $w(y) = (0, \cdots, 0)$ for all $w \in S$. This result (*) is extended to finite algebraic extensions $K = F(z)$, S being replaced by the prolongations of all $w \in S$ to K. Furthermore the imbedding of F (respectively K) in fields of formal meromorphic functions $\bar{k}((t_1, \cdots, t_n)) = L$, $\bar{k}$ an algebraic completion of k, is studied. For K the sufficient conditions r_i holomorphic in L, $w(r_m) = (0, \cdots, 0)$, discriminant of the defining equation $z^m + \sum_{i=m-1}^{0} r_i z^i = 0$ with w-value $(0, \cdots, 0)$, yield that w is unramified for K/F, and that K can be imbedded in $\bar{k}((t_1, \cdots, t_n))$. A simple example of the author establishes that result (*) also is valid for certain proper subsets of S.

O. F. G. Schilling (Chicago, Ill.)

ALGEBRAIC GEOMETRY

See also 2019.

1970:

Tallini, Giuseppe. Una proprietà grafica caratteristica della superficie di Veronese negli spazi finiti. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 19–23, 135–138.

In $S_{5,q}$ (i.e. 5-dimensional projective space over a Galois field of order $q = p^h$, p being an odd prime) the tangent planes of a Veronese surface intersect by pairs, but no three are concurrent. Being in one-one correspondence with the points of a plane, their number is $q^2 + q + 1$.

Conversely it is shown in the first paper that given in $S_{5,q}$ any set $\mathfrak{T}$ of $q^2 + q + 1$ planes, of which every two and no three meet in a point, there are $q^2 + q + 1$ points (this aggregate is denoted by $\mathfrak{F}$) through each of which passes one plane of $\mathfrak{T}$, and each plane of $\mathfrak{T}$ contains one point of $\mathfrak{F}$ (called its point of contact); that every hyperplane which contains two planes of $\mathfrak{T}$ contains precisely $q+1$ of them; that the intersections by pairs of these $q+1$ planes are in a plane π; and that the $q+1$ planes meet π in lines which form a conic envelope, the point of contact of each such line being the point of contact of the corresponding plane with $\mathfrak{F}$.

The second paper completes the proof that $\mathfrak{F}$ is in fact a Veronese surface, the planes $\mathfrak{T}$ its tangent planes, and the planes such as π those of the conics on $\mathfrak{F}$. It is also shown that this is the only family of $k \geqq q^2 + q + 1$ planes in $S_{r,q}$ ($r \geqq 5$) of which every two and no three meet in a point.

P. Du Val (London)

1971:

Guggenheimer, H. An n-dimensional analogue of the Cremona-Clebsch theorem. Bull. Amer. Math. Soc. **65** (1959), 113–116.

Let T be a Cremona transformation between two complex n-dimensional projective spaces P and P' and let M and M' be the varieties formed by the exceptional elements of T in P and P' respectively. These varieties are both of (complex) dimension $n-1$, and the relations among their homology invariants are studied in the present paper. It is shown that the Betti numbers of M and M' in any dimension are equal and that in all (real) even dimensons the torsion coefficients of M and M' are also equal. To express the relations among the torsion coefficients in (real) odd dimensions, a new numerical character $\tau_k(V)$ of a variety V in P is defined as the greatest common divisor of the orders of the k-dimensional components of the intersection of V with a generic $(k+1)$-dimensional linear subspace of P. It is shown that the torsion coefficients of M and M' in the odd dimension $2r-1$ are equal if $\tau_r(M) = \tau_r(M')$, but that if $\tau_r(M) \neq \tau_r(M')$ then $\tau_r(M')$ appears as a torsion coefficient of M and $\tau_r(M)$ appears as a torsion coefficient of M' in the dimension $2r-1$.

H. T. Muhly (Iowa City, Iowa)

1972a:

Barsotti, Iacopo. On Witt vectors and periodic group-varieties. Illinois J. Math. **2** (1958), 99–110.

1972b:

Barsotti, Iacopo. Corrections to the paper "On Witt vectors and periodic group-varieties". Illinois J. Math. **2** (1958), 608–610.

These two papers have to be read in conjunction with each other, since the first one contains (both in its statements and in its proofs) some essential mistakes for which corrections are offered in the second one. The final result may be stated as follows. Let p be the characteristic; call W_n, for each $n > 0$, the additive group of Witt vectors of length n (considered as an algebraic group-variety of dimension n, defined over the prime field). Say that a group-variety is of type (W) if it is isogenous (over the universal domain) to a product of varieties W_{n_i}. Then a commutative algebraic group is of type (W) if and only if all its elements are of finite order. If the dimension of a commutative algebraic group is n, and if its generic elements are exactly of order p^n, then the group is isogenous to W_n. Counterexamples are provided for more precise statements, included in the first paper. As usual, isogeny over the universal domain may be replaced by isogeny over any algebraically closed groundfield. The proofs depend on earlier papers by the same author.

A. Weil (Princeton, N.J.)

1973:

★**Serre, Jean-Pierre. Groupes algébriques et corps de classes.** Publications de l'institut de mathématique de l'université de Nancago, VII. Hermann, Paris, 1959. 202 pp. 3000 francs.

This is mainly an exposition of the theory of generalized Jacobian varieties by Rosenlicht [Ann. of Math. (2) **59** (1954), 505–530; **66** (1957), 80–88; MR **15**, 823; **19**, 579] and the class field theory over function fields by Lang [ibid. **64** (1956), 285–325; Bull. Soc. Math. France **84** (1956), 385–407; MR **18**, 672; **19**, 578].

Very little knowledge (mainly the sheaf-theoretic definition of algebraic varieties by Serre [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953]) is preassumed, so that this will be an easy text book on these subjects.

In Chapter I, some important results in this book are prestated.

Basic results on algebraic curves (Riemann-Roch theorem, residue formula, duality theorem) are proved in Chapter II. Some further preliminary results are proved in Chapters III and IV (Chap. III: the theorem that if f is a rational map from a non-singular curve X into an

additive group G such that it is regular outside of a closed set S, then there is a module $\mathfrak{m}$ of divisors on X with support S such that, for any divisor $D=(\varphi)$ with $\varphi \equiv 1 \bmod \mathfrak{m}$, $f(D)=0$; Chap. IV: the theory of curves with singular points), and the theory of generalized Jacobian varieties is expounded in Chapter V.

The class field theory over function fields is expounded in Chapter VI.

Homological properties of abelian varieties, proved by Barsotti [Ann. Mat. Pura Appl. (4) **38** (1955), 77–119; MR **17**, 193; Rend. Circ. Mat. Palermo (2) **5** (1956), 145–169; MR **18**, 673; and the article reviewed above] and by Rosenlicht [Amer. J. Math. **80** (1958), 685–714; MR **20** #5780] are expounded. *M. Nagata* (Kyoto)

1974:

Mattuck, Arthur. The irreducibility of the regular series on an algebraic variety. Illinois J. Math. **3** (1959), 145–149.

Let V be a projective algebraic variety over an algebraically closed field k, A the Albanese variety of V, and $f: V \to A$ the canonical map of V into A, which (replacing, if necessary, V by the birationally equivalent graph of f on $V \times A$) may be supposed single-valued. Let $V(n)$ be the n-fold symmetric product of V with itself, and $F: V(n) \to A$ the map defined by $F(x_1+x_2+\cdots+x_n)=\sum f(x_i)$, the addition on the right referring to the group law on A. The inverse images $F^{-1}(a)$ of points on A represent regular series in the sense of Albanese [Ann. Scuola Norm. Sup. Pisa (2) **3** (1934), 1–26]. The author proves the following theorem. Let $\dim V=r>1$, let $q=\dim A$, and let $g(\geq q)$ be the genus of a generic curve on a normal model of V. Then: (1) when $n \geq g$ the generic $F^{-1}(a)$ is absolutely irreducible, and F is surjective; (2) if n_0 is the smallest integer for which (1) holds, then, if $n \geq n_0+q$, every $F^{-1}(a)$ is absolutely irreducible and of dimension $nr-q$. *J. A. Todd* (Cambridge, England)

1975:

Chow, Wei-Liang. The criterion for unit multiplicity and a generalization of Hensel's lemma. Amer. J. Math. **80** (1958), 539–552.

V/k is a variety over a complete discretely-valued field k and $\bar{V}/k$ is its reduction mod p. Then every simple k-rational point of $\bar{V}$ has a foreimage in the reduction which is a simple k-rational point of V (generalized Hensel's lemma). The proof is elementary, and gives also the criterion of multiplicity one, phrased roughly: if $\bar{X}$ and $\bar{Y}$ are transversal at $\bar{a}$, there exists in $X \cdot Y$ a unique $\alpha \to \bar{a}$, and $i(\alpha, X \cdot Y)=1$. Non-elementary generalization, using multiplicity theory of local rings, to analogous statement asserting preservation of intersection multiplicities under reduction mod p at a proper intersection of $\bar{X}$ and $\bar{Y}$, together with uniqueness and rationality of the foreimage cycle. *A. P. Mattuck* (Cambridge, Mass.)

LINEAR ALGEBRA

See also 2128, 2351, 2352, 2353, 2448, 2541.

1976:

Pearl, Martin. On normal and EP_r matrices. Michigan Math. J. **6** (1959), 1–5.

A complete matrix A is called an EP_r matrix if for every linear relation between its rows, with complex coefficients α_i, the corresponding relation between the columns, with conjugate coefficients $\bar{\alpha}_i$, is satisfied, and conversely. The author establishes a number of equivalent versions of this definition. The definition is extended to matrices over an arbitrary field F, conjugacy being replaced by any involutory automorphism. It is shown that A is EP_r if A is normal and of the same rank as AA^* ($A^*=\bar{A}'$ if $\alpha \to \bar{\alpha}$ is the automorphism in F). Conditions for a matrix A to be normal in F are then established: 1. $A^*=NA=AN$ where N is non-singular; N may be chosen unitary. 2. Let K be a field containing F and the roots of A; let r_β be the rank of $A-\beta I$, $\beta \in K$. A is normal if also $(A-\beta I)(A-\beta I)^*$ has the rank r_β. There is a polynomial $f(\lambda)$ such that $A^*=f(A)$. *H. Schwerdtfeger* (Montreal, P.Q.)

1977:

Parodi, Maurice. Sur la localisation des valeurs caractéristiques des matrices et application aux polynomes. Bull. Sci. Math. (2) **81** (1957), 32–37.

All eigenvalues of a square matrix $A=(a_{ij})$ of order n lie in the union of the n disks $\{z: |a_{ii}-z| \leq r_i\}$ $(i=1, \cdots, n)$, where $r_i=\sum_1^n |a_{ij}|-|a_{ii}|$ (Geršgorin's theorem). If one disk is disjoint from the others, it contains exactly one eigenvalue of A [A. Brauer's theorem; for proofs of these results see, for example, M. Parodi, *Sur quelques propriétés des valeurs caractéristiques des matrices carrées*, Mémor. Sci. Math., no. 118, Gauthiers-Villars, Paris, 1952; MR **14**, 236]. Let $C=(c_i\delta_{ij})$ be a diagonal matrix with all $c_i>0$. By applying the above theorems to $H=C^{-1}AC$ and then applying Hölder's inequality to the sum

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|(c_jc_i^{-1}),$$

the author derives other localizations of the Geršgorin type. A numerical example shows that eigenvalues of A can sometimes be isolated by applying Brauer's theorem to H, though not by applying it to A. {Reviewer's remark: Such ideas have been stated by O. Taussky [Aeronaut. Res. Council, Rep. no. 10508, 1947].}

By letting A be the companion matrix of a polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$, the author establishes bounds for the zeros of f. In particular, if $|a_1|>1+n\alpha>2+\alpha$, then $n-1$ zeros lie in the closed unit disk, and the other zero lies in the disk with center $-a_1$ and radius

$$\tfrac{1}{2}[|a_1|-(|a_1|^2-4n\alpha)^{1/2}]<1.$$

G. E. Forsythe (Stanford, Calif.)

1978:

Lanczos, C. Linear systems in self-adjoint form. Amer. Math. Monthly **65** (1958), 665–679.

The decomposition theorem $A=X\Lambda\bar{Y}$ for an arbitrary $n \times m$ matrix A is derived from the well-known diagonalization of a symmetric matrix. Here Λ is a $p \times p$ positive diagonal matrix ($p=$rank (A)) and X and Y are $n \times p$ and $m \times p$ orthogonal matrices ($\bar{X}X=I$, $\bar{Y}Y=I$). The $(m+n)$-dimensional vectors obtained by adjoining the columns of Y to the corresponding columns of X are the eigenvectors corresponding to the p positive eigenvalues of the symmetric matrix

$$S=\begin{pmatrix} 0 & A \\ \bar{A} & 0 \end{pmatrix}.$$

The compatibility and uniqueness of the system $Ay=b$ is considered from the point of view of the above decomposition.

The extension of these ideas to function spaces is discussed. *H. F. Weinberger* (College Park, Md.)

1979:

Krull, Wolfgang. Über eine Verallgemeinerung der Hadamardschen Ungleichung. Arch. Math. **9** (1958), 42–45.

Let (g_{ik}) $(i,k=1,\ 2,\ \cdots,\ n)$ be a positive definite Hermitian matrix, and let $\mathfrak{r}=\{\rho_1,\ \cdots,\ \rho_r\}$, $\mathfrak{s}=\{\sigma_1,\ \cdots,\ \sigma_s\}$ and $\mathfrak{t}=\{\tau_1,\ \cdots,\ \tau_t\}$ be mutually disjoint subsets of $\{1,\ 2,\ \cdots,\ n\}$. The principal minor $\det(g_{\lambda\mu})$ $(\lambda,\mu\in\mathfrak{r})$ is denoted by $|\mathfrak{r}|$, with the convention that $|\mathfrak{t}_0|=1$ if $\mathfrak{t}_0$ is the empty set. Finally, put

$$\left|\begin{matrix}\mathfrak{r},\ \mathfrak{s}\\ \mathfrak{t}\end{matrix}\right| = |\mathfrak{r}\cup\mathfrak{s}\cup\mathfrak{t}|\ |\mathfrak{t}|\ |\mathfrak{r}\cup\mathfrak{t}|^{-1}\ |\mathfrak{s}\cup\mathfrak{t}|^{-1}$$

$(\mathfrak{r},\ \mathfrak{s}\neq\mathfrak{t}_0)$. The author then proves that $0<\left|\begin{matrix}\mathfrak{r},\ \mathfrak{s}\\ \mathfrak{t}\end{matrix}\right|\leqq 1$. This result generalizes the well-known inequality $\det(g_{ik})\leqq g_{11}g_{22}\cdots g_{nn}$ and similar relations.

W. Ledermann (Manchester)

1980:

Faddeev, D. K. On certain sequences of polynomials which are useful for the construction of iteration methods for solving of systems of linear algebraic equations. Vestnik Leningrad. Univ. **13** (1958), no. 7, 155–159. (Russian. English summary)

Eine Folge von Polynomen $f_n(t)$ $(n=0,\ 1,\ 2,\ \cdots)$ definiert durch die rekurrenten Formeln

$$f_0(t)=1,\quad f_1(t)=t,$$

$$f_n(t)=(1+\alpha_n)tf_{n-1}(t)-\alpha_n f_{n-2}(t)\quad (n\geqq 2)$$

wird untersucht. Drei Sätze, die sich auf die Eigenschaften dieser Funktionenfolge beziehen, werden aufgestellt. Die Funktionenfolge wird dann auf die Lösung eines Systems von linearen algebraischen Gleichungen, das in der Matrixform

$$X=BX+F$$

gegeben ist, angewandt. Dabei bezeichnet B eine reelle symmetrische Matrix, deren Eigenwerte alle im offenen Intervall $(-1,\ 1)$ liegen. *T. P. Anđelić* (Belgrade)

ASSOCIATIVE RINGS AND ALGEBRAS

1981:

Tominaga, Hisao. A note on conjugates. Math. J. Okayama Univ. **7** (1957), 75–76.

Let V be a ring with identity 1; $U=V_n$ the complete ring of $n\times n$ matrices over V, with matrix units $\{e_{ij}|i,j=1,\ \cdots,\ n\}$; $u=\sum_{i,j=1}^{n}e_{ij}c_{ij}$, $c_{ij}\in V$, $i,j=1,\ \cdots,\ n$, an element of U; D an infinite division subring of V, with $1\in D$. The author proves a theorem equivalent to the following: The set of conjugates $\{t^{-1}ut|t\in D_n=\sum_{i,j=1}^{n}e_{ij}D\}$ has cardinality either 1, or equal to that of D_n. This extends: (1) a corollary to a theorem of G. Toyoda and A. Hattori [J. Math. Soc. Japan **6** (1954), 262–265; MR **16**, 669]; (2) a theorem of I. N. Herstein [Proc. Amer. Math. Soc. **7** (1956), 1021–1022; MR **18**, 557]; (3) W. R. Scott's sharpening of (2) [ibid. **8** (1957), 303–305; MR **18**, 788]; (4) a theorem of F. Kasch [Math. J. Okayama Univ. **6** (1957), 131–133; MR **19**, 9]. The present paper is self-contained, and the theorem is proved with notable economy of effort. The author neatly dispatches the case $n=1$ with a simple computation like one of R. Brauer [Bull. Amer. Math. Soc. **55** (1949), 619–620; MR **10**, 676]. The case $n>1$ falls into two categories: (i) $c_{pq}d\neq dc_{pq}$ for some $d\in D$ and some c_{pq}. Then $\{d^{-1}ud|d\in D\}$ has cardinality = card D_n by the case $n=1$; (ii) c_{ij} commutes with all $d\in D$, $i,j=1,\ \cdots,\ n$, but $c_{pq}\neq 0$ for some $p\neq q$. Then he shows that $\{(1+de_{pq})^{-1}u(1+de_{pq})|d\in D\}$ has cardinality = card D_n.

{A paper [Canad. J. Math. **10** (1958), 374–380] of the reviewer contains fairly complete references to some related papers and results.}

C. C. Faith (University Park, Pa.)

1982:

Moriya, Mikao; and Nagahara, Takasi. On generating elements of Galois extensions of division rings. II. Math. J. Okayama Univ. **7** (1957), 89–94.

Let K be a division ring which is finite-dimensional and Galois over a division subring L. If Q is a subring of a ring R, $V_R(Q)$ denotes the centralizer of Q in R. A result of F. Kasch states: (1) K/L has a two-element generation in the sense that $K=L[k,\ h]$ is the subring generated by L, and suitable $k,\ h\in K$. In Part I [Math. J. Okayama Univ. **6** (1957), 181–190; MR **19**, 382; see the review for background and references not given here], the second-named author established, inter alia: (2) that the k and h in (1) can be taken to be conjugates, $h=u^{-1}ku$ for suitable $u\in K$. {The authors' reference to (2) in § 1 may (unintentionally) mislead the unwary reader to suppose that the result (1) also originated in I.} Now, in the present article, the authors show: (3) that the k and h in (1) can be chosen to be L-conjugates, that is, that the u of (2) can be taken to lie in $V_K(L)$; and (4) in case K/L is not simply generated, the u of (2) can be chosen to lie in $V_K(V_K(L))\cap V_K(L)$ if and only if $L\nsubseteqq V_K(L)$. Several results stating conditions under which K/L is simply generated are rederived in the paper reviewed below.

C. C. Faith (University Park, Pa.)

1983:

Nagahara, Takasi. On generating elements of Galois extensions of division rings. III. Math. J. Okayama Univ. **7** (1957), 173–178.

The notation of the preceding review will be assumed. An intermediate ring of K/L is a ring satisfying $K\supseteqq D\supseteqq L$. (Note that each intermediate ring is a division ring.) D/L is called normal in case $x^{-1}Dx=D$ for every nonzero $x\in V_K(L)$. In I the author established that $D=L[u,\ d^{-1}ud]$ for suitable $u,\ d\in D$, if either of the following two conditions hold: (*) $V_K(L)$ is commutative; (★) D/L is normal. The present program is to obtain conditions under which D/L is simple, that is, $D=L[u]$ for suitable $u\in D$. A major development is: (A) Condition (*) actually implies that D/L is simple. Furthermore, the following necessary and sufficient conditions are obtained: (B) Assuming (★), D/L is simple if and only if $L\nsubseteqq V_D(D)$. {It is known that (★) holds if and only if either $D\supseteqq V_K(L)$, or

$D \subseteqq V_K(V_K(L))$.} In particular (B) implies: (C) K/L is simple if and only if L is not contained in the center of K.

The author concludes the paper with an example showing that in order that K/L be simple it is not necessary that every intermediate ring be simple over L. A significance of this example is that the well-known theorems on minimal bases for intermediate fields of simple field extensions cannot be extended to finite-dimensional, simple, non-commutative (but Galois) division ring extensions [see Steinitz, *Algebraische Theorie der Körper*, de Gruyter, Berlin-Leipzig, 1930; p. 73, II, and p. 126, Der Lürothsche Satz]. (Note that the result (A) gives a sufficient condition in order that a minimal basis exist for every intermediate ring over L.) Furthermore, one can apply directly a result of the reviewer [Canad. J. Math. **10** (1958), 374–380; p. 379, Corollary 2] to the extension of the example just cited to see that such extensions can contain infinitely many (isomorphic) intermediate rings. (See Steinitz' "Satz der Zwischenkörper" [op. cit., p. 74, III] in this connection.)

{Reviewer's notes: (i) The author introduces some unexplained notation $E = J(\mathfrak{E}, D)$ in the proof of Lemma 4 which in context reads "E is the Galois subring of D corresponding to the automorphism group $\mathfrak{E}$ of D". (ii) The case $[L : Z] < \infty$ of Lemma 3 is simply a rederivation of Corollary 5 (2) in Part I. (iii) Although the fact that K/L of the author's example mentioned above is simple is an immediate consequence of (C), for some reason not clear to the reviewer the author appeals to a result in I to establish this.} *C. C. Faith* (University Park, Pa.)

1984:

Cohn, P. M. Rings of zero-divisors. Proc. Amer. Math. Soc. **9** (1958), 909–914.

L'auteur appelle 0-anneau tout anneau commutatif avec élément 1 tel que tout x différent de 1 est diviseur de 0; par exemple un anneau de Boole. Répondant à une question de Kaplansky, on voit qu'il existe effectivement des 0-anneaux qui ne sont pas des anneaux de Boole et plus généralement que tout anneau R (commutatif, avec élément unité) peut être immergé en un anneau S en sorte que tout élément, non diviseur de 0, de S est un élément inversible de R. De plus si R est une algèbre sur un corps F, S peut être alors considéré comme une algèbre sur F.

Pour construire l'anneau S précédent on procède par induction transfinie, associant d'une manière générale à tout anneau R et un $a \in R$ (a non inversible en R) un anneau R' en sorte que $aa' = 0$ pour un a' non nul de R' et que tout élément inversible de R' est un élément de R, inversible en R. On pose $R_{n+1} = R_n'$ pour tout n fini et on passe aux transfinis quelconques.

On en déduit que toute algèbre R sur un corps F, telle que les seuls éléments inversibles de R sont les éléments non nuls de F, peut être plongée en une 0-algèbre S (par définition une algèbre sur F dont tous les éléments non contenus en F sont des diviseurs de 0). De plus R est produit direct de corps K_t [au sens de McCoy, *Rings and ideals*, Open Court Publ. Co., La Salle, Ill., 1948; MR **10**, 96; le radical de Jacobson de R est nul], chaque K_t étant une extension de K. De plus chaque élément x de $R - F$ est transcendant sur F, sauf si F est le corps à deux éléments et x idempotent. Enfin si R est de dimension finie et $R \neq F$, alors R est une algèbre de Boole.

J. Guerindon (Rennes)

1985:

Roquette, Peter. Abspaltung des Radikals in vollständigen lokalen Ringen. Abh. Math. Sem. Univ. Hamburg **23** (1959), 75–113.

This is an expository article on Cohen's structure theorem of complete local rings [Trans. Amer. Math. Soc. **59** (1946), 54–106; MR **7**, 509], along the line of the proofs by Geddes and Narita (the former [J. London Math. Soc. **29** (1954), 334–341; MR **16**, 213] proved the equal characteristic case only and the latter [J. Math. Soc. Japan **7** (1955), 435–443; MR **18**, 6] completed the proof of the general case). *M. Nagata* (Kyoto)

1986:

McLaughlin, J. E. A note on regular group rings. Michigan Math. J. **5** (1958), 127–128.

In this note the author gives elementary proofs for the following theorems concerning group rings which were also established by the reviewer using homological methods [Proc. Amer. Math. Soc. **8** (1957), 658–664; MR **19**, 390]. Theorem 1: Let A be a ring with unit, G a group and R the group ring of G over A. If R is regular (in the sense of von Neumann), then G is a torsion group and A is uniquely divisible by the order of each element in G. Theorem 2: If G is a locally finite group and A is a regular ring with unit which is uniquely divisible by the order of every element in G, then the group ring of G with respect to A is regular. *M. Auslander* (Waltham, Mass.)

1987:

Curtis, Charles W. Quasi-Frobenius rings and Galois theory. Illinois J. Math. **3** (1959), 134–144.

Let R be a quasi-Frobenius ring, i.e., a ring with unit satisfying both minimum conditions which is injective when regarded as a left-module over itself. Let M be a faithful finitely generated projective left R-module. (Faithfulness in this context means that M has direct summands isomorphic to Re for each primitive idempotent e in R.) The author then shows that if $C = \mathrm{Hom}_R(M, M)$, M is again a projective (faithful) right C-module and C is a quasi-Frobenius ring. An example is given to show that the result is no longer true if M is not projective. As is pointed out by the author in a remark "added in proof", these results are contained in a paper by Morita [Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142; MR **20** #3183]. By combining results of the author's with some of Nesbitt and Thrall [Ann. of Math. (2) **47** (1946), 551–567; MR **8**, 64] it then also follows that $\mathrm{Hom}_C(M, M) = R$. The stage is now set for an application of these results to the Galois theory of simple rings with minimum conditions [Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Providence, R.I., 1956; MR **18**, 373]. Let L be the ring of all linear transformations on a finite-dimensional vector space M over a division ring D, G a group of automorphisms of L, $U(G)$ the ring of endomorphisms of M generated by D and the semilinear transformations that induce the elements of G, and $I(G) = \mathrm{Hom}_{U(G)}(M, M)$ the fixed ring of G. The author then gives necessary and sufficient conditions ensuring that $U(G)$ and $I(G)$ be quasi-Frobenius and M a faithful finitely generated projective $U(G)$-module. The usual Galois correspondence between complete groups G, with $U(G)$ quasi-Frobenius and M a projective finitely generated $U(G)$-module, and

a proper subset of all quasi-Frobenius subrings R of L, with M a projective finitely generated R-module, is then established. The author closes by generalizing the usual concepts of height and index for $I(G)$ in L and gives a relation between these and the reduced order of G.

A. Rosenberg (Evanston, Ill.)

1988:

Goldie, A. W. The structure of prime rings under ascending chain conditions. Proc. London Math. Soc. (3) **8** (1958), 589–608.

A prime ring is one in which $IJ \neq 0$ for any two-sided ideals I, J of the ring. The author establishes the structure of prime rings with ascending chain condition. The fullest form of his theorem is the following: A ring R has a right and left quotient ring which is a complete matrix ring $M_n(D)$, where D is a division ring, if and only if (i) R is a prime ring; (ii) R has ascending chain condition for right and left annihilators; (iii) every direct sum of non-zero left or right ideals in R has a finite number of terms. The proof depends on the development and application of the properties of uniform ideals. A right ideal U is called uniform if $U \neq 0$ and if I, J non-zero right ideals contained in U implies $I \cap J \neq 0$. An element of a ring is said to be right uniform if the principal right ideal generated by it is uniform. The maximal uniform right ideals are called basic right ideals. For a subset S of the ring, let S_r [S_l] be the right [left] annihilator of S. A right ideal I is said to be relatively complemented if $I \neq 0$ and there exists a non-zero right ideal J such that $I \cap J = 0$. For any such right ideal I, the maximal such J's are the relative complements of I and are called complement right ideals. By applying a theorem of O. Ore [Ann. of Math. (2) **32** (1931), 463–477], the author first proves that a ring without zero divisors satisfying the ascending chain condition for right ideals has a right quotient division ring (th. 1). All definitions and results given for right ideals hold for left ideals with suitable modifications.

Assume, now, that R is a ring satisfying conditions (i), (ii), and (iii) of the main theorem. The author next develops the properties of uniform ideals of R. It is proved that R is itself a uniform right ideal if and only if R has no zero divisors, and in this case R is also a uniform left ideal and has a right quotient division ring which is also a left quotient ring (th. 2). In this case R is called a uniform ring. For $u \in R$, u is left uniform if and only if u is right uniform; and if R is not a uniform ring then for any uniform right ideal U, $U_l \neq 0$ (th. 3). If U is a uniform right ideal and $\mathscr{E}(U)$ is the ring of R-endomorphisms of U, then $\mathscr{E}(U)$ has no zero divisors, $\theta \in \mathscr{E}(U)$ implies θ non-singular, and $\mathscr{E}(U)$ has a right quotient division ring (th. 4). If B is a basic right ideal of R then $B_{lr} = B$; the basic right ideals coincide with the minimal annihilator right ideals; the right ideals u_r, for u uniform, coincide with the maximum annihilator right ideals (th. 5).

Although it is not true, as it is with minimal ideals, that sums of uniform ideals can always be reduced to direct sums, the direct sums of uniform ideals play a key role. The length of a direct sum is the number of terms in the direct sum. Direct sums of basic right ideals of maximum length have the same length (th. 6). This unique maximum length is called the right dimension of R and the right and left dimensions of R are equal (th. 7). It is then shown that the maximal complement right ideals coincide with the maximal annihilator right ideals (th. 8) and the complement right ideals coincide with the proper annihilator right ideals (th. 9).

An element of R is said to be regular if it is not a zero divisor. A right ideal I of R has a non-zero intersection with each non-zero right ideal of R if and only if I contains regular elements (th. 10). By making use of a known necessary and sufficient condition that a ring with regular elements have a quotient ring [Jacobson, *The theory of rings*, Amer. Math. Soc., New York, 1943; MR **5**, 31; pp. 118–119], the author proves that R has a right quotient ring which is also a left quotient ring (th. 11). If Q is the quotient ring of R then Q is isomorphic to the complete matrix ring $M_n(D)$, where n is the dimension of R and D is the quotient division ring of endomorphisms of any uniform right ideal of R (th. 12). In order to complete the proof of the main theorem (th. 13), it is shown that conditions (i), (ii), and (iii) hold for any subring of $M_n(D)$ which has $M_n(D)$ as its right and left quotient ring.

D. W. Wall (Ann Arbor, Mich.)

1989:

Saitô, Tôru. Note on the distributive laws. Amer. Math. Monthly **66** (1959), 280–283.

A w-ring is a system $(R, +, \cdot)$ where $(R, +)$ is an abelian group, $(R, \cdot)$ is a semi-group, and there exist elements e_1, e_2 of R such that

$$(1) \qquad \begin{aligned} a\cdot(b+c) &= a\cdot b + a\cdot c - e_1 \\ (a+b)\cdot c &= a\cdot c + b\cdot c - e_2 \end{aligned}$$

for all a, b, c in R. It is immediate that $e_1 = 0\cdot 0 = e_2$. $e = 0\cdot 0$ is called the defining element of the w-ring. It is easy to see that $a\cdot 0 = e = 0\cdot a$ and $a\cdot e = e = e\cdot a$ for all a in R. The author shows that there exists a 1-1 correspondence between w-rings $(R, +, \cdot)$ with defining element e and rings $(R, +, *)$ with annihilator e. In fact, $a\cdot b - a * b = e$ for all a, b in R.

A related system is the c-ring obtained by replacing (1) in the definition of a w-ring by

$$(1') \qquad (a+b)\cdot(c+d) = a\cdot c + a\cdot d + b\cdot c + b\cdot d$$

for all a, b, c, d in R. c-rings are shown to be w-rings with defining elements e such that $3e = 0$. The existence of c-rings that are not rings and the use of the property $a\cdot 0 = e = 0\cdot a$ for all a, respectively, provide solutions to the two parts of a problem given by the reviewer [Problem 4784, Amer. Math. Monthly **65** (1958), 289], which was inspired by an inaccuracy on page 18 of John L. Kelley's *General topology* [Van Nostrand, Toronto-New York-London, 1955; MR **16**, 1136].

D. W. Jonah (Providence, R.I.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 2004, 2019.

1990:

Harris, Bruno. Centralizers in Jordan algebras. Pacific J. Math. **8** (1958), 757–790.

The definition of the centralizer of a Jordan subalgebra is based on the concept of operator commutativity (x, y operator commute if $R_xR_y = R_yR_x$). Operator commutativity for Jordan algebras is studied, and results of

N. Jacobson [Proc. Amer. Math. Soc. **3** (1952), 973–976; MR **14**, 531] are extended to algebras over fields of characteristic not two. It is shown that if the enveloping associative algebra of a special Jordan algebra J contains no central nilpotent elements, then the centralizer of a subalgebra of J is just the set of elements commuting in the associative multiplication with the elements of the subalgebra. This implies that some later results can be regarded as generalizations of the associative algebra results if the associative algebras are converted into Jordan algebras by using the Jordan product $x \cdot y = xy + yx$.

Part of the theory of a single linear transformation in a finite-dimensional vector space is generalized to the subalgebra generated by a single element in a simple finite-dimensional Jordan algebra. Such an algebra is equal to the centralizer of its centralizer. A formula of Frobenius giving the dimension of the centralizer of a single linear transformation in terms of the degrees of its invariant factors is generalized to elements in any central simple Jordan algebra. The last two results mentioned are proved by studying the various simple Jordan algebras one at a time and getting information about the invariant factors of an element x which are defined by the author and about the centralizer of the algebra generated by x. Much of the effort here is devoted to the case where x is nilpotent. Analogues of the centralizer and double centralizer theorems for simple finite-dimensional subalgebras of the associative algebra of all linear transformations on a vector space [Rosenberg, Math. Z. **61** (1954), 150–159; MR **16**, 563] are found to hold for simple finite-dimensional Jordan subalgebras of the Jordan algebra of all self-adjoint linear transformations on a self-dual vector space (possibly infinite dimensional). Some of Rosenberg's results are generalized from the class of rings of all continuous linear transformations on a vector space to primitive rings with minimal ideals. *L. A. Kokoris* (Chicago, Ill.)

1991:

Beaumont, Ross A. Generalized rings. Proc. Amer. Math. Soc. **9** (1958), 876–881.

Let R be an additive (abelian) group closed under a binary multiplication. Given two integers $m \geq 2$, $n \geq 2$, if for any $a_1, \cdots, a_m, b_1, \cdots, b_n \in R$ the equation $(\sum a_i)(\sum b_j) = \sum a_i b_j$ holds, then R is called an (m, n)-distributive ring. Any ring (in the usual sense) is (m, n)-distributive for all m, n, but an (m, n)-distributive ring need not be a ring. In fact a necessary and sufficient condition for an (m, n)-distributive ring to be a ring is that $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$. More generally, if (0) is the two-sided ideal generated by 0 in an (m, n)-distributive ring R, then $R^* = R - (0)$ is a ring and any proper homomorphic image of R is a homomorphic image of R^*. The author also gives conditions for R to split over (0).

P. M. Cohn (Manchester)

1992:

Ree, Rimhak. The simplicity of certain nonassociative algebras. Proc. Amer. Math. Soc. **9** (1958), 886–892.

Starting with an elementary p-group V of order p^n $(n > 1)$, and two alternating biadditive functions $\phi(u, v)$, $\psi(u, v)$ $(u, v \in V)$ taking values in a field K of characteristic p and satisfying certain conditions, the author constructs by means of its multiplication table a non-associative algebra L of dimension $p^n - 2$ over K. He shows that L is simple if $p > 2$. If special conditions are satisfied L can be a Lie algebra, and can be a special case of the Lie algebras considered by R. Block ["New simple Lie algebras of prime characteristic", to appear in Trans. Amer. Math. Soc.]. It is mentioned that these Lie algebras are all included in a more general class of Lie algebras of characteristic p, for which a construction is given in terms of derivations of the group algebra of V over K.

I. M. H. Etherington (Edinburgh)

1993:

Kokoris, Louis A. On rings of (γ, δ)-type. Proc. Amer. Math. Soc. **9** (1958), 897–904.

Among the algebras of type (γ, δ) [Albert, Portugal. Math. **8** (1949), 23–36; MR **11**, 316; Kokoris, Canad. J. Math. **8** (1956), 250–255; MR **17**, 1050] those of type $(-1, 0)$ and $(1, 1)$ are residual cases in the classification of almost alternative algebras relative to quasi-equivalence. In this paper the following results concerning rings are proved. (a) Any ring of type $(-1, 0)$ is anti-isomorphic to a ring of type $(1, 1)$. (b) Let A be a simple ring of type $(1, 1)$ whose characteristic is not 2 and which contains an idempotent e; then either A is associative, or e is the unity element of A. In an appendix is given the extension by Kleinfeld, to rings, of the author's earlier results on algebras of type (γ, δ).

R. D. Schafer (Cambridge, Mass.)

1994:

Albert, A. A. Finite noncommutative division algebras. Proc. Amer. Math. Soc. **9** (1958), 928–932.

The author provides a method of constructing finite nonassociative division algebras and, in particular, the first known class of such algebras of characteristic two.

Briefly, the construction is as follows: Let K be a field of degree n over a field F of $q = p^m$ elements and denote the identity element of K by e. Assume that $q > 2$, $n > 2$ and select any element c of K such that $c \neq 1$, $c \neq a^{q-1}$ for any a of K. Let $R(x)$ denote the right translation, $a \to ax$, of K (as a vector space over F) and denote by S the generating automorphism, $x \to xS = x^q$, of the Galois group of K over F. Now define linear transformations

$$P^{-1} = I - SR(c), \quad Q^{-1} = S - R(c)$$

of K and a new multiplication by

$$x \cdot y = (xP)(yQS) - c(xPS)(yQ).$$

Under this multiplication the vector space K is a non-associative division algebra $D(K, c)$ with identity element $f = e - c$. *L. J. Paige* (Los Angeles, Calif.)

1995a:

Hsieh, Pang-chieh. Chains of admissible subgroups of groups with operators. Acta Math. Sinica **7** (1957), 631–640. (Chinese. English summary)

1995b:

Hsieh, Pang-chieh. Chains of admissible subgroups of groups with operators. Sci. Sinica **7** (1958), 704–715. [Two versions of the same article.]

The author proves that if G is a group with left operator domain Ω, then the Ω-subgroups of G satisfy the ascending [descending] chain condition, or ACC [DCC], if and only if

G contains normal Ω-subgroups $N_1, \cdots, N_n$ such that (a) the ACC [DCC] is satisfied by the Ω-subgroups of G/N_i, $i=1, \cdots, n$, and (b) the ACC [DCC] is satisfied by the Ω-subgroups of $\bigcap_{i=1}^{n} N_i$. This is applied in the obvious way to obtain a necessary and sufficient condition for the left ideals of a nonassociative ring or of a near-ring to satisfy the ACC [DCC]. Then from some simple results on chain conditions in direct products of operator groups, the author shows that the left ideals of the total matric algebra R_n over a nonassociative ring R satisfy the ACC [DCC] if and only if the left ideals of R satisfy the ACC [DCC]. Finally he discusses briefly the extension of this work to normal chains and to two-sided operator groups and ideals. *W. E. Deskins* (East Lansing, Mich.)

HOMOLOGICAL ALGEBRA

See 1987, 2236.

GROUPS AND GENERALIZATIONS

See also 1992, 1995a-b, 2065, 2202, 2485.

1996:

★**Hall, Marshall, Jr. The theory of groups.** The Macmillan Co., New York, N.Y., 1959. xiii+434 pp. $8.75.

The complete table of contents reads as follows.

1. Introduction. 1.1 Algebraic laws. 1.2 Mappings. 1.3 Definitions for groups and some related systems. 1.4 Subgroups, isomorphisms, homomorphisms. 1.5 Cosets. Theorem of Lagrange. Cyclic groups. Indices. 1.6 Conjugates and classes. 1.7 Double cosets. 1.8 Remarks on infinite groups. 1.9 Examples of groups.

2. Normal subgroups and homomorphisms. 2.1 Normal subgroups. 2.2 The kernel of a homomorphism. 2.3 Factor groups. 2.4 Operators. 2.5 Direct products and cartesian products.

3. Elementary theory of abelian groups. 3.1 Definition of abelian group. Cyclic groups. 3.2 Some structure theorems for abelian groups. 3.3 Finite abelian groups.

4. Sylow theorems. 4.1 Falsity of the converse of the theorem of Lagrange. 4.2 The three Sylow theorems. 4.3 Finite p-groups. 4.4 Groups of orders p, p^2, pq, p^3.

5. Permutation groups. 5.1 Cycles. 5.2 Transitivity. 5.3 Representation of a group by permutations. 5.4 The alternating group A_n. 5.5 Intransitive groups. Subdirect products. 5.6 Primitive groups. 5.7 Multiply transitive groups. 5.8 On a theorem of Jordan. 5.9 The wreath product. Sylow subgroups of symmetric groups.

6. Automorphisms. 6.1 Automorphisms of algebraic systems. 6.2 Automorphisms of groups. Inner automorphisms. 6.3 The holomorph of a group. 6.4 Complete groups. 6.5 Normal or semi-direct products.

7. Free groups. 7.1 Definition of free group. 7.2 Subgroups of free groups. The Schreier method. 7.3 Free generators of subgroups of free groups. The Nielsen method.

8. Lattices and composition series. 8.1 Partially ordered sets. 8.2 Lattices. 8.3 Modular and semi-modular lattices. 8.4 Principal series and composition series. 8.5 Direct decompositions. 8.6 Composition series in groups.

9. A theorem of Frobenius; solvable groups. 9.1 A theorem of Frobenius. 9.2 Solvable groups. 9.3 Extended Sylow theorems in solvable groups. 9.4 Further results in solvable groups.

10. Supersolvable and nilpotent groups. 10.1 Definitions. 10.2 The upper and lower central series. 10.3 Theory of nilpotent groups. 10.4 The Frattini subgroup of a group. 10.5 Supersolvable groups.

11. Basic commutators. 11.1 The collecting process. 11.2 The Witt formulae. The basis theorem.

12. The theory of p-groups; regular p-groups. 12.1 Elementary results. 12.2 The Burnside basis theorem. Automorphisms of p-groups. 12.3 The collection formula. 12.4 Regular p-groups. 12.5 Some special p-groups. Hamiltonian groups.

13. Further theory of abelian groups. 13.1 Additive groups. Groups modulo one. 13.2 Characters of abelian groups. Duality of abelian groups. 13.3 Divisible groups. 13.4 Pure subgroups. 13.5 General remarks.

14. Monomial representations and the transfer. 14.1 Monomial permutations. 14.2 The transfer. 14.3 A theorem of Burnside. 14.4 Theorems of P. Hall, Grün, and Wielandt.

15. Group extensions and cohomology of groups. 15.1 Composition of normal subgroup and factor group. 15.2 Central extensions. 15.3 Cyclic extensions. 15.4 Defining relations and extensions. 15.5 Group rings and central extensions. 15.6 Double modules. 15.7 Cochains, coboundaries, and cohomology groups. 15.8 Applications of cohomology to extension theory.

16. Group representation. 16.1 General remarks. 16.2 Matrix representation. Characters. 16.3 The theorem of complete reducibility. 16.4 Semi-simple group rings and ordinary representations. 16.5 Absolutely irreducible representations. Structure of simple rings. 16.6 Relations on ordinary characters. 16.7 Imprimitive representations. 16.8 Some applications of the theory of characters. 16.9 Unitary and orthogonal representations. 16.10 Some examples of group representations.

17. Free and amalgamated products. 17.1 Definition of free product. 17.2 Amalgamated products. 17.3 The theorem of Kurosch.

18. The Burnside problem. 18.1 Statement of the problem. 18.2 The Burnside problem for $n=2$ and $n=3$. 18.3 Finiteness of $B(4, r)$. 18.4 The restricted Burnside problem. Theorems of P. Hall and G. Higman. Finiteness of $B(6, r)$.

19. Lattices of subgroups. 19.1 General properties. 19.2 Locally cyclic groups and distributive lattices. 19.3 The theorem of Iwasawa.

20. Group theory and projective planes. 20.1 Axioms. 20.2 Collineations and the theorem of Desargues. 20.3 Introduction of coordinates. 20.4 Veblen-Wedderburn systems. Hall systems. 20.5 Moufang and Desarguesian planes. 20.6 The theorem of Wedderburn and the Artin-Zorn theorem. 20.7 Doubly transitive groups and near-fields. 20.8 Finite planes. The Bruck-Ryser theorem. 20.9 Collineations in finite planes.

There is a bibliography of about 125 items, an index, and an index of special symbols.

We shall point out some special features of the book, using the numbering of the table of contents: The wreath product and its application to the symmetric groups (5.9) is in no other text. The same is true of the material appropriate to the Burnside problem (11, 12.3, 18). Nor does any other text do justice to P. Hall's theory of regular

p-groups (12.4). The theory of ordinary representations appears (in part) in Burnside [W. Burnside, *Theory of groups of finite order*, 2nd ed., Cambridge Univ. Press, 1911; Dover, New York, 1955; MR **16**, 1086] and (for characteristic zero) in Speiser [A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, 3rd ed., Springer, Berlin, 1927; Dover, New York, 1945; MR **7**, 410], but in none of the more recent texts on abstract group theory except the present one (16); as a refreshing consequence, the author is able (in 16.9) to prove two famous theorems (due to Burnside and Frobenius, respectively) which most authors state without proof. The only prior attempt to include non-Desarguesian geometry and combinatorics in a book on group theory was that by Carmichael [R. Carmichael, *Introduction to the theory of groups of finite order*, Ginn, Boston, 1937; Dover, New York, 1956; MR **17**, 823]; the present chapter 20 is much more effective and forms a worthy companion to Pickert [G. Pickert, *Projektive Ebenen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 399]. Although the present volume has much to say about infinite groups, the emphasis is more on finite groups than in either Kuroš [A. G. Kuroš, *The theory of groups*, Chelsea, New York, 1955, 1956; MR **17**, 124; **18**, 187] or Specht [W. Specht, *Gruppentheorie*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956; MR **18**, 188].

The only criticisms which occur to the reviewer are relatively minor. The bibliography does not compare with that of Kuroš (compare the similar remark in the review of Specht).—The definition of basic commutators (and more particularly, of the weight of a basic commutator) seems open to question. Is (in 11.1) a basic commutator a group element or not? If the answer is yes, the weight of a basic commutator is not well defined. If the answer is no, there are pitfalls in the use of basic commutators which the author avoids skilfully but does not point out to the reader.—There are misprints, but apparently none of importance.

By some inexplicable trick of memory, the reviewer, in reading this book, recaptured all the excitement of his first reading of Burnside—an excitement which Burnside no longer affords. He recommends the volume with enthusiasm. *R. H. Bruck* (Madison, Wis.)

1997:

★**Baumgartner, Ludwig. Gruppentheorie.** Dritte, vollständig neubearbeitete Aufl. Sammlung Göschen Bd. 837. Walter de Gruyter and Co., Berlin, 1958. 110 pp. (3 inserts) DM 2.40.

This little book cannot be expected to cover the same ground as a treatise selling at 15 or 20 times its price. Rather, it introduces the student to various useful concepts in group theory, illustrates these with simple examples, proves the more basic elementary theorems, and offers 94 exercises, the solutions to which occupy pp. 100–107. For example, here is exercise 74 (p. 71): Prove: Under any automorphism, a subgroup always goes into a subgroup (especially, $\mathfrak{E}$ into $\mathfrak{E}$). As another indication: § 27 (pp. 53–54), entitled "Simple and Hamiltonian groups", contains the relevant definitions, states (without proof, but with reference to a suitable treatise) that the alternating groups of degree at least 5 are simple, states (without proof) that both finite and infinite Hamiltonian groups exist, and exhibits the quaternion group.

On the one hand, the book is a good bargain. On the other hand, the prospective purchaser should not expect too much from the following (somewhat abbreviated) description of the contents. I. Introduction to the group concept (sets, algebraic operations, definition of finite and infinite groups). II. Group-theoretic concepts and methods (cyclic groups, subgroups, coset expansions). III. On finite groups (group tables, permutations, intersections and products). IV. Permutability of elements and subgroups (normalizers, conjugate classes, the centre, theorems on normal divisors, simple and Hamiltonian groups, centralizers, direct products). V. Factor groups. VI. Homomorphy (the standard elementary theorems). VII. Automorphy (automorphism groups, inner automorphisms, the holomorph). VIII. Endomorphy; characteristic and fully invariant subgroups (also the commutator subgroup). IX. Free groups and groups with relations between elements (reduced words, free groups, generators and relations). X. Series of groups (normal series; the theorems of Zassenhaus, Schreier and Jordan-Hölder). XI. A closer look at the postulates (German: Genaueres über die Gruppenpostulates). Solutions of the exercises. Index. Table I: The symmetric group of degree 3. Table II: The dihedral group. *R. H. Bruck* (Madison, Wis.)

1998:

Adyan, S. I. On algorithmic problems in effectively complete classes of groups. Dokl. Akad. Nauk SSSR **123** (1958), 13–16. (Russian)

In this note the author indicates further cases of insoluble algorithmic problems for finitely presented (f.p.) groups. They concern the discrimination problem for a group property α and a class K of f.p. groups, i.e. the task of finding an algorithm to decide whether or not a given group of K has the property α. A class K of f.p. groups is said to be effectively complete if there exists an algorithm $\mathfrak{A}$ that converts the presentation of an arbitrary f.p. group G into a presentation of some group of K with a subgroup isomorphic to G. Such an algorithm $\mathfrak{A}$ is said to leave a group G_0 fixed if it converts every presentation of G_0 into another presentation of G_0. In each of the following three cases K is an effectively complete class of f.p. groups, α is an invariant group property, and it is assumed that K contains at least one group with the property α, but that there exist groups that cannot be embedded in any f.p. group with the property α. The conclusion in each time that the discrimination problem for α and K is algorithmically insoluble: (1) The algorithm $\mathfrak{A}$ leaves at least one group with the property α fixed; in particular, (2) the algorithm $\mathfrak{A}$ leaves every group of K fixed; (3) K is recursive, for every f.p. group it can be verified algorithmically whether or not it belongs to K. An example of a recursive and effectively complete class of f.p. groups is the f.p. groups that coincide with their derived group.

The author also shows in this note how the proof of a fundamental lemma in his earlier work [same Dokl. **117** (1957), 9–12; Trudy Moscov. Mat. Obšč. **6** (1957), 231–298; MR **20** #2371, #2370] can be substantially simplified.

K. A. Hirsch (London)

1999:

Lyndon, R. C. The equation $a^2b^2=c^2$ in free groups. Michigan Math. J. **6** (1959), 89–95.

R. Vaught conjectured that in a free group the equation

$a^2b^2=c^2$ implies $ab=ba$. The author uses a technique developed by Nielsen [Math. Ann. **79** (1918), 269–272] to prove this conjecture, but he points out that the method does not enable one to decide whether the equation $a^3b^3=c^3$ in a free group implies $ab=ba$. {This question has recently been answered affirmatively by G. Baumslag, using a direct method; see a forthcoming paper in the J. London Math. Soc.} *P. M. Cohn* (Manchester)

2000:

Schützenberger, Marcel Paul. Sur l'équation $a^{2+n}=b^{2+m}c^{2+p}$ dans un groupe libre. C. R. Acad. Sci. Paris **248** (1959), 2435–2436.

If a, b, c are elements of a free group such that $a^n=b^mc^p$, where $n, m, p\geqq 2$, then $bc=cb$ and a, b, c are all contained in a cyclic subgroup. The proof outlined here, using the theory of semigroups, may be regarded as essentially a "cancellation argument". {The reviewer [see review above] obtained results containing the case $n=m=p=2$, a problem suggested by R. Vaught; in addition to the present author, G. Baumslag, E. Schenkman [Ann of Math. (2) **70** (1959), 562–564], and J. Stallings have independently and by different methods obtained generalizations containing the case $n=m=p\geqq 2$.} It is noted also that $a^2=b^2c^2d^2$ has non-trivial solutions, and that $a^n=b^{-1}c^{-1}bc$ for $n\geqq 2$ implies $a=1$.

R. C. Lyndon (Ann Arbor, Mich.)

2001:

Bryant, Steven. Isomorphism order for Abelian groups. Pacific J. Math. **8** (1958), 679–683.

Let k be a finite cardinal or $\aleph_0$. A group G is said to have isomorphism order k if any group H is embeddable in G provided that, in the case of a finite k, every subgroup of H generated by at most k elements is embeddable in G, and, in the case of $k=\aleph_0$, every finitely generated subgroup of H is embeddable in G. It is proved that an abelian group G has isomorphism order k if and only if G is the direct sum of less than k copies of the additive group of rationals and of fewer than k subgroups of the rationals mod 1. *A. Kertész* (Debrecen)

2002:

Pic, Gheorghe. Sur le quasi-centre d'un groupe. II. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 467–472.

For the definition and simplest properties of the quasi-centre of a group see the author's preceding paper [Acad. R. P. Romîne. Stud. Cerc. Mat. **4** (1953), 7–21; MR **16**, 11]. In this note the author proves the following results: 1. If the quasi-centre is non-abelian, the group must be periodic. 2. The factor-group of the quasi-centre may be infinite cyclic (in contrast to the centre and to Baer's nucleus). But this cannot happen when the quasi-centre is non-abelian. For the factor-group of the quasi-centre to be infinite cyclic it is necessary that the quasi-centre should be abelian and for at least one prime number its primary component should contain elements of arbitrarily high order. 3. If the group is periodic and the factor-group of the quasi-centre is cyclic, then the group is the direct product of its primary components. 4. If the group is the direct product of its primary components, so is the quasi-centre. *K. A. Hirsch* (London)

2003:

McLain, D. H. Finiteness conditions in locally soluble groups. J. London Math. Soc. **34** (1959), 101–107.

Notation: max, min, $\max_N$, $\min_N$ denote the maximal and minimal condition for subgroups and normal subgroups, respectively; FG stands for finitely generated; a group G is n-step polynilpotent if it has a series of normal subgroups $G={}^0N\supseteq {}^1N\supseteq\cdots\supseteq {}^nN=1$ with ${}^iN/{}^{i+1}N$ nilpotent (of finite class). It is known [see e.g. P. Hall, Proc. London Math. Soc. (3) **4** (1954), 419–436; MR **17**, 344] that for a soluble group $\max_N$ implies FG. The author begins with a neat example to show that here "soluble" cannot be replaced by "locally soluble". He then proceeds to determine the class of locally soluble groups for which $\max_N$ implies FG. The main result of this section is that a group G satisfying $\max_N$ is soluble if and only if it is locally n-step polynilpotent for some fixed natural number n. For the case $n=1$ see the author's earlier paper [Proc. Cambridge Phil. Soc. **52** (1956), 5–11; MR **17**, 456] and Gluškov [Mat. Sb. (N.S.) **30 (72)** (1952), 79–104; MR **13**, 908]. In particular, for a locally supersoluble group (supersoluble in the sense of Baer [Proc. Amer. Math. Soc. **6** (1955), 16–32; MR **16**, 671]: every homomorphic image has a cyclic normal subgroup) the properties $\max_N$, max, and FG coincide. A companion result is that in a locally supersoluble group $\min_N$ implies min. Now a locally soluble group with min (even for abelian subgroups only) is soluble [see Černikov, Dokl. Akad. Nauk SSSR **65** (1949), 21–24; MR **10**, 590], so that in a locally supersoluble group $\min_N$ implies solubility. The author ends with an example to show that here "supersoluble" cannot be replaced by "soluble". *K. A. Hirsch* (London)

2004:

Plotkin, B. I. Algebraic sets of elements in groups and Lie algebras. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 133–138. (Russian)

A group with maximal condition is called an M-group, and a group with a local system of M-groups is called an LM-group or a group satisfying the condition LM. The author gives a short proof of the following theorem due to Baer [Math. Z. **66** (1957), 341–363; MR **19**, 12]. Theorem 1: In any group G the subgroup generated by two normal LM-subgroups again satisfies LM. The proof uses the notion of an algebraic set in a group, defined as follows: If E is a finite subset of a group G and $x\in G$, denote by $K(x, E)$ the subgroup of G generated by x and all the commutators $[\cdots[[x, a], b], \cdots, c]$, where $a, b, \cdots, c\in E$. If $K(x, E)$ is finitely generated, E is said to be x-algebraic, and E is algebraic if it is x-algebraic for all $x\in G$. Now the proof of Theorem 1 is based on the following two remarks: (i) A finite algebraic subset of a normal subgroup H of G is algebraic in G; (ii) if H is a normal LM-subgroup of G and $E=E^{-1}$ is a finite algebraic set in G such that the subgroup K generated by E is an M-group and $KH=G$, then G is an LM-group.

Theorem 1 has the following analogue for Lie algebras (taking an M-algebra to be a Lie algebra of finite dimension). Theorem 2: In any Lie algebra the subalgebra generated by two ideals satisfying LM again satisfies LM [cf. Levitzki, Trans. Amer. Math. Soc. **74** (1953), 384–409; MR **14**, 720 for the analogue for associative rings]. This allows one to define the LM-radical in any Lie algebra A as the union $LM(A)$ of all ideals satisfying LM; by

Theorem 2, $LM(A)$ always satisfies LM, but $A/LM(A)$ need not be LM-semisimple (i.e. its LM-radical need not be 0). However, as the author shows, if A is an algebraic Lie algebra, then $A/LM(A)$ is LM-semisimple. This is derived from Theorem 3: If an algebraic Lie algebra A has an ascending normal series with LM-factors, then A satisfies LM. Here algebraic sets in a Lie algebra are defined in analogy with the algebraic sets in a group and a Lie algebra A is algebraic if every one-element subset of A is algebraic. *P. M. Cohn* (Manchester)

2005:

Ghika, Al. La notion de base dans certains groupes à opérateurs. Rev. Math. Pures Appl. **1** (1956), no. 3, 59–72.

L'auteur généralise la notion de base d'un module par rapport à un anneau en l'étendant au cas d'un groupe G non-abélien à opérateurs; il démontre que G admet une base satisfaisant au théorème d'échange; la représentation des elements de G n'est plus unique en général. Ensuite l'auteur étudie des espaces dits quasivectoriels, c'est-à-dire des modules par rapport à un "corpoïde" [voir Ghika, Com. Acad. R. P. Romîne **6** (1956), 239–243; MR **18**, 638], en y définissant des bases spéciales appelées surbases qui satisfont encore au théorème d'échange; il démontre que chaque base est contenue dans une surbase.

G. K. Kalisch (Minneapolis, Minn.)

2006:

Schenkman, Eugene; and Wade, L. I. The mapping which takes each element of a group onto its nth power. Amer. Math. Monthly **65** (1958), 33–34.

Let $\tilde{n}$ be the mapping of each element g of a group G onto g^n, and let $G(n)$ and $G\{n\}$ denote respectively the subgroups of G generated by nth powers and by elements whose orders divide n. In an Abelian group, $\tilde{n}$ is an endomorphism. The authors prove conversely, that (1) if $\tilde{n}$ is an endomorphism, then so is $\overline{1-n}$, and the subgroup $G(n^2-n)$ and factor group $G/G\{n^2-n\}$ are Abelian; (2) if $\tilde{n}$ is an automorphism, then $G \cong G(n)$ and $G(n-1)$ is in the center of G, so $G/G\{n-1\}$ is Abelian. The group G is Abelian if $G\{n^2-n\}=1$ in case (1), or if $G\{n-1\}=1$ in case (2). *J. S. Frame* (East Lansing, Mich.)

2007:

Dubisch, Roy. A chain of cyclic groups. Amer. Math. Monthly **66** (1959), 384–386.

Let $\mathfrak{G}_0$ be a cyclic group of order m, and $\mathfrak{G}_i$ the automorphism group of $\mathfrak{G}_{i-1}$ $(i=1, 2, \cdots)$. The author determines those integers m for which all the groups $\mathfrak{G}_i$ are cyclic. The remarks in the final sentence need some qualification: the quoted result of Miller is true for finite Abelian groups but not true in general.

P. J. Higgins (London)

2008:

Sprague, Roland. Über teilerfremde Paare imprimitiver Gruppen. Math. Nachr. **18** (1958), 222–225.

For $i=1, 2$, let E_i denote a partititon of the first n natural numbers into b_i sets of a_i integers $(n=a_ib_i)$, and let G_i be the set of all permutations of $1, 2, \cdots, n$ which preserve the partititon E_i. The problem is to construct E_1, E_2 so that G_1, G_2 have only the identity permutation in common. Using graph theory, the author produces an example with $n=36$, $a_1=3$, $a_2=2$. He gives a new proof for an old example [Sprague, Math. Z. **54** (1951), 27–33; MR **12**, 845] with $n=24$, $a_1=a_2=3$, by describing the two partitions by an incidence matrix of $b_1=8$ rows and $b_2=8$ columns, consisting of zeros and ones. In conclusion, he points out that, by interchange of zeros and ones in the incidence matrix, he may obtain from his example with $n=24$ a new example with $n=40$, $a_1=a_2=5$ and, from his example with $n=36$, a new example with $n=180$, $a_1=15$, $a_2=10$. *R. H. Bruck* (Madison, Wis.)

2009:

Wielandt, Helmut. Über die Existenz von Normalteilern in endlichen Gruppen. Math. Nachr. **18** (1958), 274–280.

Let G be a finite group, H be a subgroup of G and H^* be a normal subgroup of H. Moreover, let mj and j be respectively the order of G and the index of H^* in H. The author is interested in the existence of a normal subgroup G^* of G such that (1) $H \cap G^* = H^*$, $HG^* = G$.

First he proves (1) under the assumption: (2) $H \cap H^t \subset H^*$ for every t in $G-H$ (Theorem 1). In this case, G^* is unique and is exhibited explicitly. The proof involves a neat and elementary use of the orthogonality properties of group characters (for representations over the complex field). Thus the author has given a new proof for a celebrated theorem (corresponding to the case $H^*=1$) of Frobenius [G. Frobenius, S.-B. Preuss. Akad. Wiss. Berlin **1901**, 1216–1230].

Next the author introduces the centralizer, F, of H in G and replaces (2) by (2′) $H \cap H^t \subset H^*$ for every t in $G-F$. In addition, he assumes (3) $(i, j)=1$, where i is the index of H in F. In the presence of (2′), (3), he shows that (1) is equivalent to the existence of a subgroup F^* of index i in F, such that F^* contains H^* and is normal in F. If either G^* or F^* exists, both exist uniquely (Theorem 2). Theorem 2 contains Theorem 1 as a special case, but rests on the latter.

As an application of Theorem 2, the author replaces (2′) by (2″) if t is in G, either $H=H^t$ or $H \cap H^t \subset H^*$. He also replaces (3) by (3′) $(m, j)=1$; and he adds: (4) there are precisely m solutions in G of the equation $x^m=1$. That (2″), (3′), (4) imply (1) is the content of Theorem 3, which sharpens a theorem of Feit [Proc. Amer. Math. Soc. **7** (1956), 177–187; MR **17**, 1051].

R. H. Bruck (Madison, Wis.)

2010:

Wielandt, Helmut. Eine Distributivitätseigenschaft konjugierter Untergruppen. Math. Z. **69** (1958), 345.

Let H be a subgroup of finite index n in a group G, and set $H_i=a(i)^{-1}Ha(i)$, where $Ha(1), \cdots, Ha(n)$ are the n distinct cosets. The author proves the following distributive law:

$$H_i \cap \bigcup_{j \neq k} (H_j \cap H_k) = \bigcup_{j \neq k} (H_i \cap H_j \cap H_k),$$

where i is any fixed integer chosen from $1, 2, \cdots, n$, and where j, k range over $1, 2, \cdots, n$. The proof reduces at once to the case that $H_i=H$, G is finite and H is its own normalizer in G. Moreover, it suffices to show the existence of a normal subgroup N of G such that $H \cap N=R$, $HN=G$, where R denotes the right hand side of the proposed identity. Since R is normal in G, the result

follows rapidly from recent work of the author [#2009 above]. *R. H. Bruck* (Madison, Wis.)

2011:

Gorenstein, Daniel; and Herstein, I. N. A class of solvable groups. Canad. J. Math. **11** (1959), 311–320.

Part of an investigation of finite groups G which can be expressed in the form ABA for suitable subgroups A, B. In this paper A and B are cyclic of relatively prime orders h and k. The conclusion is that G is solvable, that its Sylow subgroups are abelian or quaternion, and that its order is hkw, where w divides some power of k.

Graham Higman (Oxford)

2012:

Landin, Joseph; and Reiner, Irving. Automorphisms of the Gaussian unimodular group. Trans. Amer. Math. Soc. **87** (1958), 76–89.

Let G be the ring of Gaussian integers and G_n the group of $n \times n$ unimodular matrices over G. Denote the transpose of X by X' and the conjugate by $\bar{X}$. In the case $n=2$, let

$$T_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the elements T_0, S_0, and P_0 generate the group G_2. The authors obtain a system of generators for the automorphism group $\mathfrak{A}_n$ of the group G_n. Their main theorem asserts that $\mathfrak{A}_n$ is generated by: 1. $X \to AXA^{-1}$, $A \in G_n$; 2. $X \to X'^{-1}$ (may be omitted when $n=2$); 3. $X \to \bar{X}$; 4. $X \to (\det X)^k X$, where $k=1$ if n is even and $k=2$ if n is odd; 5. For $n=2$ only, $(P_0, S_0, T_0) \to (P_0, -S_0, -T_0)$.

C. E. Rickart (New Haven, Conn.)

2013:

Landin, Joseph; and Reiner, Irving. Automorphisms of the two-dimensional general linear group over a Euclidean ring. Proc. Amer. Math. Soc. **9** (1958), 209–216.

The automorphisms of the general linear group $\mathrm{GL}_n[R]$ of dimension n over a ring with unit have been found in case R is a field, a skew-field, the ring of rational integers, and in certain other cases [Hua and Reiner, Trans. Amer. Math. Soc. **71** (1951), 331–348; MR **13**, 328; Z. X. Wan, Sci. Sinica **7** (1958), 885–933; MR **20** #5237; and the paper reviewed above].

In the present paper the automorphisms of $\mathrm{GL}_2[R]$ are found in case R satisfies all the hypotheses: I. R is a commutative, Euclidean principal ideal domain, integrally closed in its quotient field; every element of R is a linear combination of units with (natural) integer coefficients; R has more than two units. In this case the automorphisms (a's) of $\mathrm{GL}_2[R]$ consist of inner a's; a's induced by a's of R; $A \to \lambda(\det A)A$, where λ is an endomorphism of the units U of R such that $\alpha \to \alpha\lambda[\lambda(\alpha)]$ is an automorphism of U; and a special automorphism σ needed in case 1 is not a sum of differences of units; σ transforms $X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ into $\pm X(t)$, $t \in R$.

J. L. Brenner (Palo Alto, Calif.)

2014:

Baer, Reinhold. Die Potenzen einer gruppentheoretischen Eigenschaft. Abh. Math. Sem. Univ. Hamburg **22** (1958), 276–294.

Let $\mathfrak{E}$ be a non-empty class of groups, closed under the operation of taking quotient groups, and satisfying the following condition: if G/A and G/B have $\mathfrak{E}$, then so does $G/A \cap B$. Set $\mathfrak{E}^1 = \mathfrak{E}$ and define a new group property $\mathfrak{E}^2$ thus: G has $\mathfrak{E}^2$ shall mean that every quotient group $F \neq 1$ of G has a normal subgroup $N \neq 1$ in which F induces an $\mathfrak{E}^1$-group of automorphisms. Then $\mathfrak{E}^2$ satisfies the same conditions as those assumed of $\mathfrak{E}^1$ and $\mathfrak{E}^1 \leqq \mathfrak{E}^2$. In this way one defines all finite powers $\mathfrak{E}^i$ and takes $\mathfrak{E}^\infty$ as their union. The author studies the structure of $\mathfrak{E}^\infty$-groups under suitable finiteness conditions.

If a group G has a series $1 = N_0 < N_1 < \cdots < N_k = G$ of characteristic subgroups such that N_i/N_{i-1} is upper nilpotent (i.e., hypercentral or a ZA-group), then G is said to have degree k.

(1) If G has a normal subgroup N of degree i and G/N is $\mathfrak{E}^j$ then G is $\mathfrak{E}^{i+j}$. The special case $j=1$ has a converse for $\mathfrak{N}$-groups, i.e., groups satisfying the maximal condition on normal subgroups: (1′) If G is $\mathfrak{N}$ and $\mathfrak{E}^i$, then there exists a normal subgroup N of degree $i-1$ such that G/N is $\mathfrak{E}$. It follows that (2) an $\mathfrak{N}$-group is $\mathfrak{E}^\infty$ if, and only if, it is soluble-by-$\mathfrak{E}$. (If N is normal in a group G and N has a property $\mathfrak{P}$ and G/N a property $\mathfrak{Q}$, then we say G is $\mathfrak{P}$-by-$\mathfrak{Q}$). Groups that are $\mathfrak{N}$ and $\mathfrak{E}^\infty$ can also be characterised as follows: (3) If G is $\mathfrak{N}$, then G is $\mathfrak{E}^\infty$ if, and only if, every non-trivial quotient F of G has a non-trivial normal N such that N is abelian or F induces in N an $\mathfrak{E}$-group of automorphisms. By taking $\mathfrak{E}$ to be the class of all finite groups, one obtains the following corollary [Baer, Math. Z. **66** (1956), 269–288; MR **18**, 639; p. 276]: If G is $\mathfrak{N}$ and every non-trivial quotient F of G has a non-trivial finite or abelian normal subgroup, then G is soluble-by-finite.

The author next proves two theorems involving a stronger condition than $\mathfrak{N}$: G has $\mathfrak{N}'$ means that every normal subgroup of G has $\mathfrak{N}$. (4) Every $\mathfrak{N}'$-$\mathfrak{E}^\infty$ group is torsionfree-by-torsion if, and only if, every $\mathfrak{N}'$-group which is finite soluble-by-$\mathfrak{E}$ and has no torsionfree normals is necessarily a torsion group. If $\mathfrak{E}$ is the class of all finite groups, there results the corollary [K. A. Hirsch, Proc. London Math. Soc. (2) **49** (1946), 184–194; MR **8**, 132; p. 188]: every soluble group with maximal condition on its subgroups has a torsionfree normal subgroup of finite index. (5) Every $\mathfrak{N}'$-group which is $\mathfrak{E}^\infty$-by-$\mathfrak{E}^\infty$ is itself $\mathfrak{E}^\infty$, provided every $\mathfrak{N}$-group which is $\mathfrak{E}$-by-($\mathfrak{E}$ or abelian) and has no non-trivial abelian normal is itself $\mathfrak{E}$.

The minimal condition on normal subgroups, called $\mathfrak{m}$, imposes a similar structure on $\mathfrak{E}^\infty$-groups as does $\mathfrak{N}$: (6) The $\mathfrak{m}$-group G is $\mathfrak{E}^\infty$ if, and only if, G is soluble-by-$\mathfrak{E}$. Moreover, for both $\mathfrak{m}$ and $\mathfrak{N}$, $\mathfrak{E}^{\omega+1} = \mathfrak{E}^\omega$; but this is still true if $\mathfrak{m}$ and $\mathfrak{N}$ are replaced by the class of all groups G with the following property: whenever X is a product of normals in which G induces $\mathfrak{E}^\omega$-groups of automorphisms, then G does the same in X. (8) If G has no abelian normal subgroups and contains no direct product of infinitely many normal subgroups, then if G is $\mathfrak{E}^\omega$, it must be $\mathfrak{E}$.

{Reviewer's remarks: The proof of (8) can be used to show more: If G has no abelian normal subgroups, then G is $\mathfrak{E}^2$ if, and only if, G is a direct product of groups having $\mathfrak{E}$.

A number of the author's arguments can be shortened by using the following observation: G is $\mathfrak{E}^2$ if, and only if, there exists a series $1 < N_1 < \cdots < N_\omega = G$ of normal subgroups of G with the following property: if λ is not a limit ordinal and N_λ^* is the centralizer, modulo $N_{\lambda-1}$, of

N_λ in G, then G/N_λ^* has $\mathfrak{C}$. Let $H=\bigcap N_\lambda^*$. Then G/H is residually $\mathfrak{C}$ (by construction) and, by an easy argument, H is upper nilpotent.} *K. W. Gruenberg* (London)

2015:

Ponizovskiĭ, I. S. On irreducible matrix representations of finite semigroups. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 139–144. (Russian)

Let S be a finite semigroup with zero, 0. Let n_0 be the least positive integer such that $S^{n_0+1}=S^{n_0}$, and let $J_1=S^{n_0}$. Suppose that $J_1, J_2, \cdots, J_i$ have already been defined. Let K_i be the largest 2-sided ideal in J_i. Let n_i be the least positive integer such that $K_i^{n_i+1}=K_i^{n_i}$. Then let $J_{i+1}=K_i^{n_i}$. Thus one obtains a properly decreasing sequence $S\supset J_1\supset J_2\supset\cdots\supset J_w$ $(=\{0\})$. Each J_i is proved to be a 2-sided ideal in S. Let $L_i=J_i/K_i$ $(i=1, 2, \cdots, w-1)$. The semigroups L_i are completely simple. The irreducible representations of L_i by matrices over an arbitrary field can all be extended in one and only one way to such representations of J_i; and these representations in turn to representations of S. Conversely, every irreducible matrix representation of S arises in this way. Since Clifford has shown how to obtain all irreducible matrix representations of completely simple semigroups [Amer. J. Math. **64** (1942), 327–342; MR **4**, 4], all of the irreducible matrix representations of S can be found. For explicit calculations in a special case, see Hewitt and Zuckerman [Illinois J. Math. **1** (1957), 188–213; MR **19**, 249].

E. Hewitt (Seattle, Wash.)

2016:

Peák, István. Ein Satz über Halbgruppen. Publ. Math. Debrecen **6** (1959), 111–112.

The author proves that a cancellative simple semigroup with a non-empty centre is a group.

G. B. Preston (Shrivenham)

2017:

Pollák, Georg; und Rédei, Ladislaus. Die Halbgruppen, deren alle echten Teilhalbgruppen Gruppen sind. Publ. Math. Debrecen **6** (1959), 126–130.

The authors show that a semigroup in which every proper subsemigroup is a group must be of one of the following three types: (a) a periodic group; (b) a semigroup with only two elements; (c) a semigroup generated by a single element a satisfying the relation $a^n=a^2$ for some integer $n>2$. *G. B. Preston* (Shrivenham)

2018:

Šutov, È. G. Potential divisibility of elements in semigroups. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 75–103. (Russian)

Given a semigroup S, a pair (x, z) of elements of S is said to have left divisibility in S if $z=xu$ for some $u\in S$; similarly (y, t) has right divisibility in S if $y=vt$ $(v\in S)$. Now two sets Ω_l, Ω_r of pairs of elements of S are called sets of (simultaneous) potential divisibility if S can be embedded in a semigroup S^* in which the pairs of Ω_l have left divisibility and the pairs of Ω_r have right divisibility. Denote by S^1 the semigroup obtained by adjoining a unit-element to S, then two (obvious) necessary conditions for two subsets Ω_l, Ω_r of $S\times S$ to be sets of potential divisibility are (1) $a_1x=a_2x$ implies $a_1z=a_2z$ for all $(x, z)\in\Omega_l$, $a_i\in S^1$, and (2) $tb_1=tb_2$ implies $yb_1=yb_2$ for all $(y, t)\in\Omega_r$, $b_i\in S^1$. The author proves that (1), (2) and the following condition (3) are sufficient for Ω_l, Ω_r to form sets of potential divisibility. (3) If x, t are as in (1), (2) and $ax=tb$, where $a, b\in S^1$, then there exists $c\in S^1$ with $ax=tcx$. Various corollaries are obtained by taking one of Ω_l, Ω_r to be empty or of the special form $M\times N$ (M, N subsets of S). Examples from and applications to the multiplicative semigroup of $n\times n$ matrices over a field are given, and an example is constructed to show that for a given semigroup S and given Ω_l, Ω_r these sets may separately have potential left divisibility and right divisibility respectively, without being sets of simultaneous potential divisibility [for such an example cf. also P. M. Cohn, J. London Math. Soc. **31** (1956), 169–181; MR **18**, 14]. A final result concerns central divisibility; the semigroup S is centrally divisible if, for any $x, y\in S$, there exist $s, t\in S$ such that $x=syt$. Now the author proves that every semigroup can be embedded in a centrally divisible semigroup.

P. M. Cohn (Manchester)

2019:

Tits, J. Sur la trialité et les algèbres d'octaves. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 332–350.

This paper falls naturally into several distinct but closely inter-related parts: (1) A new approach to the Bol 3-nets defined by a Moufang loop. (2) The concept of a T-geometry. (3) The principle of triality (for 6-dimensional hyperquadrics over an arbitrary field K) and an elementary proof of an allied property. (4) Triality and the Cayley numbers.—In his introduction, the author stresses (3), (4); for the purposes of this review it will be convenient to discuss (1)–(4) in order.

(1) Bol nets. {Two convenient references for this part are the reviewer's book [R. H. Bruck, *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958; MR **20** #76] and that of Pickert [G. Pickert, *Projektive Ebenen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 399].} A 3-net (German: Gewebe; French: tissu) may be regarded as a system consisting of a triple of disjoint, non-empty sets D_1, D_2, D_3 (the three classes of "lines" of the net) and of ordered triples (a_1, b_2, c_3) of elements of D_1, D_2, D_3 respectively (the "points" of the net) such that any two of a_1, b_2, c_3 may be assigned arbitrarily in their respective sets and then determine the third uniquely.

The author shows that a Bol net (or 3-net defined by a Moufang loop) may be defined as a set Δ, partitioned into three disjoint non-empty sets D_1, D_2, D_3, together with a binary operation $(\circ)$ on Δ subject to the following axioms: (i) $a\circ(a\circ b)=b$ for all a, b in Δ. (ii) $a\circ(b\circ c)=(a\circ b)\circ(a\circ c)$ for all a, b, c in Δ. (iii) $a\circ b=b\circ a$ for all a, b of Δ not belonging to the same set D_i. (iv) $D_i\circ D_j=D_i$ or D_k according as $i=j$ or i, j, k is a permutation of 1, 2, 3. —Here the points are the triples of types $(a_1, b_2, a_1\circ b_2)$, $(a_1, a_1\circ c_3, c_3)$ and $(b_2\circ c_3, b_2, c_3)$.

For each a in the Bol net $(\Delta, \circ)$, by (ii), (i), the mapping $x\to a\circ x$ is an automorphism of $(\Delta, \circ)$, called "inner". The following condition (C) is sufficient (but not necessary) in order that the group of inner automorphisms be transitive on Δ: (C) If a, b are in Δ, there exists at least one c in Δ such that $a\circ c=b\circ c=c$.

(2) T-geometries. By a T-geometry we are to understand a system $\mathcal{T}=(T_1, T_2, T_3; I)$ consisting of three non-empty sets T_1, T_2, T_3 and a symmetric binary

"incidence" relation I, defined on the union $\Theta = T_1 \cup T_2 \cup T_3$ and subject to the following axiom: (I) If i, j, k is any permutation of 1, 2, 3 and if t_i (in T_i) and t_j (in T_j) are not incident, there exists one and only one element t_k in T_k which is simultaneously incident with t_i and t_j.

An automorphism of $\mathcal{T}$ is an incidence-preserving permutation of Θ. $\mathcal{T}$ is called symmetric if each of the six permutations of the objects T_1, T_2, T_3 is induced by at least one automorphism of $\mathcal{T}$.

For i, j, k any permutation of 1, 2, 3, D_i denotes the set of all automorphisms a of $\mathcal{T}$ which map T_i upon itself, interchange T_j, T_k, and satisfy the following three conditions: (i) For each t_j in T_j, the elements p_j and $a(p_j)$ are incident. (ii) If p_i (in T_i) is simultaneously incident to p_j (in T_j) and to $a(p_j)$, then $a(p_i) = p_i$. (iii) a^2 is the identity. Then, for all a, b in $\Delta = D_1 \cup D_2 \cup D_3$, the element aba is in Δ and is denoted by $a \circ b$; moreover, $(\Delta, \circ)$ is a Bol net, in the sense of (1), provided at least two of D_1, D_2, D_3 are non-empty.—The author gives a simple example of a symmetric T-geometry, in terms of an arbitrary non-empty set T and an arbitrary symmetric binary relation J on T, for which the associated Bol net is the 3-net of the group of all permutations of T which preserve J.

(3) Triality. K being an arbitrary field, a "hyperquadric" will mean a 6-dimensional, non-degenerate hyperquadric of maximal index defined over K.

Let Q_1 be a hyperquadric. The V_3's (i.e., linear 3-dimensional varieties) contained in Q_1 fall into two systems and can be represented by the points of two hyperquadrics Q_2, Q_3 in such a manner that the V_3's of a system which contain a given point of Q_1 are represented by the points of a V_3 of the corresponding hyperquadric. The relation between Q_1, Q_2, Q_3 is perfectly symmetric and is known as a triality relation.

Let Q_1, Q_2, Q_3 be three hyperquadrics in a triality relation and let P_1, P_2, P_3 be the respective 7-dimensional projective spaces containing them. Denote by T_i the set of points (over K) of Q_i. Then $\mathcal{T} = (T_1, T_2, T_3; I)$ is a T-geometry if I is defined as follows: for $i \neq j$, the points p_i (in T_i) and p_j (in T_j) are incident if the V_3 of Q_i corresponding to p_j contains p_i; and points p_i, q_i in the same set T_i are incident if and only if they coincide. By use of a theorem of Chow [Ann. of Math. **50** (1949), 32–67; MR **10**, 396] the author is able to show that $\mathcal{T}$ defines a Bol net $(\Delta, \circ)$ in the manner of (2). Moreover, if i, j, k is any permutation of 1, 2, 3 and if a is any element of D_i, the automorphism $x \to a \circ x$ of $(\Delta, \circ)$ (see (1)) may be extended to a permutation of the set $P_1 \cup P_2 \cup P_3$ made up of a harmonic homology of P_i, a collineation of P_j upon P_k, and the inverse collineation of P_k upon P_j.

By translating these and further results into the language of loop theory, the author obtains the following: Let P be a 7-dimensional projective space (over K), let Q be a (6-dimensional) hyperquadric in P, and let D be the set of points of P not belonging to Q. Then D possesses the natural structure of a Moufang loop $(D, *)$, determined to within a projectivity preserving Q. If a is any element of D, each of the permutations $x \to a * x$, $x \to x * a$ and $x \to x^{-1}$ of D may be extended to a projectivity of P.

(4) The Cayley numbers. (French: l'algèbre d'octaves.) Now the author assumes that the hyperquadrics of the second paragraph of (3) have no points over K. The theorems of (3) remain true in a sharper form: in the second paragraph of (3), replace D_i by P_i; in the third paragraph, replace D by P. The author goes on to determine precise conditions on the quadratic forms defining the Q_i. These forms can be taken to be norm forms, related by certain conditions, of (8-dimensional) alternative division algebras over K. In particular, $(D, *)$ turns out to be the multiplicative loop (modulo the elements of norm one) of an alternative division algebra over K.

R. H. Bruck (Madison, Wis.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 2228.

2020:

Helmberg, Gilbert. A theorem on equidistribution on compact groups. Pacific J. Math. **8** (1958), 227–241.

Let G be a compact group, and $g_1, \cdots, g_n$ be n elements of G. Generalizing results of Eckmann [Comment. Math. Helv. **16** (1943/44), 249–263; MR **6**, 146] and Hlawka [Rend. Circ. Mat. Palermo (2) **4** (1955), 33–47; MR **17**, 594], the author shows: There exists a universal way of arranging the multiple sequence $\{g_1^{i_1} \cdots g_n^{i_n} : 0 \leq i_k < \infty, k = 1, \cdots, n\}$ into a single sequence $\{g_r\}_1^\infty$, such that $\{g_r\}_1^\infty$ is equidistributed in G, provided that for each non-trivial irreducible representation R of G there exists a k with $1 \leq k \leq n$, such that $R(g_k)$ does not have 1 as eigenvalue. A similar theorem is proved about a closed subgroup H of G. For abelian groups (but not in general) the main condition of the theorem is necessary and sufficient for equidistribution. Analogues for finite groups are discussed; the exponents i_k are then restricted to $0 \leq i_k <$ order of g_k.

H. Samelson (Ann Arbor, Mich.)

2021:

Auslander, L.; and Auslander, M. Solvable Lie groups and locally euclidean Riemann spaces. Proc. Amer. Math. Soc. **9** (1958), 933–941.

In this paper, the authors determine explicitly the following two classes of groups: (1) connected and simply-connected solvable Lie groups which act transitively on a torus with discrete isotropic subgroups; (2) the fundamental groups of compact, locally Euclidean Riemannian manifolds which are homeomorphic with the coset spaces of connected solvable Lie groups.

H. C. Wang (Evanston, Ill.)

2022:

Kristensen, Leif. Invariant metrics in coset spaces. Math. Scand. **6** (1958), 33–36.

It is proved that if H is a compact subgroup of the group G, then G/H satisfies the first axiom of countability if and only if G/H has an invariant metric. If M is metrizable, and G is generated by a compact neighborhood, there is an invariant metric for M such that a subset is compact if and only if it is closed and bounded.

P. S. Mostert (New Orleans, La.)

FUNCTIONS OF REAL VARIABLES

See also 2176.

2023:

Lipiński, J. S. Sur les fonctions approximativement continues. Colloq. Math. **5** (1958), 172–175.

A. Denjoy, al quale si deve la definizione di funzione reale approssimativamente continua [Bull. Soc. Math. France **43** (1915), 161–248], aveva dimostrato che ogni funzione siffatta, se limitata, è una derivata. In questo lavoro si dimostra che condizione necessaria è sufficiente affinchè la funzione $f(x)$ (finita o no) sia approssimativamente continua, e che tutte le funzioni

$$f_a^b(x) = \max\{a, \min[b, f(x)]\}$$

(qualunque siano i due numeri reali, a, b con $a<b$) siano delle derivate. *T. Viola* (Torino)

2024:

Marcus, S. Sur les fonctions dérivées, intégrables au sens de Riemann et sur les dérivées partielles mixtes. Proc. Amer. Math. Soc. **9** (1958), 973–978.

The author first proves the identity of the concept of 'neighborliness' due to W. W. Bledsoe [same Proc. **3** (1952), 114–115; MR **13**, 634] and that of 'quasicontinuity' due to S. Kempisty [Fund. Math. **19** (1932), 184–197]. He deduces in Theorem 2 that a derivative which is continuous almost everywhere on (a, b) is quasicontinuous on (a, b), but in later theorems gives a negative reply to the natural suggestions that the condition of neighborliness may be sufficient for a bounded derivative to be Riemann-integrable or that every bounded derivative is neighborly. He concludes with some remarks on the commutativity of partial derivation. *U. S. Haslam-Jones* (Oxford)

2025:

Sunyer Balaguer, F. On the determination of a function by its derived numbers. Collect. Math. **10** (1958), 185–194. (Spanish)

Proofs of the theorems announced in an earlier paper by the author [C. R. Acad. Sci. Paris **245** (1957), 1690–1692; MR **19**, 946]. *T. A. Botts* (Charlottesville, Va.)

2026:

Tideman, M. Comments on an old inequality. Nordisk Mat. Tidskr. **6** (1958), 27–28, 56. (Swedish. English summary)

The inequality in question is

$$\sum_{n=1}^{\infty} n(n^2+x^2)^{-2} < \tfrac{1}{2}x^{-2} \quad (x \neq 0).$$

It was published by Mathieu in 1890 without proof. The first proof was given in 1952 by L. Berg [cf. also Van der Corput and Heflinger, Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 15–20; MR **17**, 949]. The author proves by simple methods the more general inequality

$$kx^2 \sum_{n=1}^{\infty} n^{2k-1}(n^2+x^2)^{-k-1} < \tfrac{1}{2} \quad (x \neq 0,\ k = 1, 2, \cdots).$$

D. J. Hofsommer (Amsterdam)

MEASURE AND INTEGRATION

See also 1955, 2020, 2152.

2027:

Croft, H. T. Note on a plane measure problem. Mathematika **5** (1958), 125–127.

Where l is a given number between zero and one, consider the class of all sets A lying inside the unit circle in the plane such that for no straight line does the linear Lebesgue measure of the intersection of A with this line exceed $2l$. It is shown that as A ranges over this class the supremum of the plane Lebesgue measure of A is $2\sin^{-1} l + 2l(1-l^2)^{1/2}$ and that this supremum is not attained. *T. A. Botts* (Charlottesville, Va.)

2028:

Vinti, Calogero. Una ripartizione del continuo ed una osservazione sulle funzioni continue rispetto ad una e non misurabili rispetto ad un'altra variabile. Rend. Sem. Mat. Univ. Padova **27** (1957), 253–266.

Let $E=(0, 1)$ and $Q=E\times E$. Using the well-known construction of Vitali, the author shows that there are a sequence of disjoint sets $E_n \subset E$, and three functions f, g, h defined on Q, such that: (a) $E=\bigcup_1^\infty E_n$, $m_i(\bigcup_1^N E_n)=0$ for any N; (b) $f(x, y)$ is measurable in x and converges to $f(x, 0)$ for all x, if $y\to 0$, but for any set $S=\{x\}$, $|S|>0$, $f(x, y)$ does not converge uniformly on S; (c) g and h are bounded, continuous in y and not measurable in x; (d) given $\varepsilon>0$, there is a set $S=\{x\}$ such that $|E-S|<\varepsilon$ and g is uniformly continuous in y on $S\times E$; (e) for any set $S=\{x\}$, $|S|>0$, h is not uniformly continuous in y on $S\times E$. *M. Cotlar* (Buenos Aires)

2029:

Zamansky, Marc. Construction des espaces L^p par complétion. J. Math. Pures Appl. (9) **38** (1959), 97–116.

In this paper, the author gives a detailed exposition of an integration theory which he recently announced in two notes [C. R. Acad. Sci. Paris **244** (1957), 2882–2885, 3009–3011; MR **19**, 640]. In addition to this he shows how the Lebesgue spaces L^p, $p\geq 1$ can be introduced by completion. *W. A. J. Luxemburg* (Pasadena, Calif.)

2030:

Mišík, Ladislav. Einige Bemerkungen zur Mass- und Integraltheorie. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 81–102. (Slovak. Russian and German summaries)

The author is concerned with measure spaces such as (X, R, μ), and with "integral spaces" (X, F, I), where F denotes a vector space of functions defined on X, and I a positive homogeneous additive functional. The last spaces are called complete if $f\in F$, $|g|\leq|f|$, $I(|f|)=0$ implies $g\in F$. Complete extensions and σ-extensions are studied and conditions are formulated that guarantee the uniqueness of such extensions. As an example of other results we quote the following: For $i=1, 2$ let (X_i, F_i, I_i) be complete and σ-complete and belong to (X_i, R_i, μ_i) in the usual sense. Then $(X_1, F_1, I_1)=(X_2, F_2, I_2)$ if for all $A\in R_i$, $\mu_i(A)<\infty$, there exist $B_1, B_2\in R_j$, $j\neq i$, such that $B_1\subset A\subset B_2$, $\mu_j(B_1)=\mu_i(A)=\mu_j(B_2)$.

František Wolf (Berkeley, Calif.)

2031:

Nakanishi, Shizu. L'intégrale (E.R.) et la théorie des distributions. Proc. Japan Acad. **34** (1958), 565–570.

Relations entre la théorie de l'intégration dans les espaces rangés et la théorie des distributions.

A. Appert (Angers)

2032:

Okano, Hatsuo. (ER)-integral of Radon-Stieltjes type. Proc. Japan Acad. **34** (1958), 580–584.

Étude de l'intégrale (ER) de K. Kunugi du type de Radon-Stieltjes sur un espace à mesure abstraite.

A. Appert (Angers)

2033:

Okano, Hatsuo. Multiplication of (ER)-integrable functions. Proc. Japan Acad. **34** (1958), 585–586.

Étude d'un cas où le produit de deux fonctions (ER) intégrables est aussi (ER) intégrable. Intégration par parties des fonctions (ER) intégrables.

A. Appert (Angers)

2034:

Régnier, André. Un théorème ergodique ponctuel purement ensembliste. C. R. Acad. Sci. Paris **248** (1959), 2700–2701.

Énoncé, sans démonstrations, d'un "théorème purement ensembliste qui généralise le théorème ergodique de Birkhoff", puis de quelques "conséquences métriques".

2035a:

Kolmogorov, A. N. A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 861–864. (Russian)

2035b:

Kolmogorov, A. N. Entropy per unit time as a metric invariant of automorphisms. Dokl. Akad. Nauk SSSR **124** (1959), 754–755. (Russian)

In the first of these papers the author attempts to define the theory of a dynamical system (automorphism and flow) as an invariant under metric isomorphism (conjugacy) and to use this invariant to exhibit examples of dynamical systems (automorphisms and flows) with a continuous spectrum which are spectrally isomorphic (all having a Lebesgue spectrum of multiplicity $\aleph_0$) but which are not isomorphic metrically, thus solving in the negative an important problem in the theory of dynamical systems of some years standing.

There is however a serious mistake in the theorem implying that the entropy as defined in this paper is a metric invariant (theorem 2 is wrong) and the second paper deals with a modification of the definition of entropy of automorphisms which ensures its metric invariance. The examples of automorphisms given in the first paper are shown by use of the modified notion of entropy to have the spectral and metric properties for which they have been constructed in the first place. As regards the examples of flows given in the first paper see the review of Ya. Sinaĭ's papers below. The notions of conditional quantity of information and entropy appearing in the first paper have no direct bearing on the problem of conjugacy once the modifications mentioned above were made but may be of independent interest.

Let (X, S, μ) be a Lebesgue space [cf. Rohlin, Uspehi Mat. Nauk (N.S.) **4** (1949), no. 2 (30), 57–128; MR **11**, 40] of total measure 1, and let $\{S_t\}$ denote a flow or automorphism (when t takes only integral values) on X. If $\mathfrak{A}_0$ is a closed subalgebra of S, let $\mathfrak{A}_t = S_t\mathfrak{A}_0$ and $\mathfrak{A}_{s,t} = \mathbf{V}_{s<u\leq t}\mathfrak{A}_u$, $S(\mathfrak{A}_0) = \mathbf{V}_t\mathfrak{A}_t$ where $\mathbf{V}$ denotes the algebra which is the common refinement of the algebras following it. The flow (or automorphism) is called quasi regular if it satisfies the following condition: There exists a closed subalgebra $\mathfrak{A}_0$ of S such that (i) $\mathfrak{A}_t \subseteq \mathfrak{A}_{t'}$ if $t \leq t'$, (ii) $\mathbf{V}_t\mathfrak{A}_t = S$, (iii) $\mathbf{\Lambda}_t\mathfrak{A}_t = N$, where N is the trivial algebra $(\emptyset, X)$. The author states that it follows from the work of Plessner [Dokl. Akad. Nauk SSSR **23** (1939), 327–330; **25** (1939), 710–712; MR **1**, 338] that a quasi-regular system has a homogeneous Lebesgue spectrum and that in the case of an automorphism the multiplicity of the spectrum is $\aleph_0$.

Let L be a closed subalgebra of S. According to Rohlin [cf. reference above] L generates a decomposition of X, $X = \bigcup C$, and a canonical system of measures μ_C. For $x \in C$ and $A \in S$ let $\mu_x(A|L) = \mu_C(A \wedge C)$. If $\mathfrak{A}$, $\mathfrak{B}$, L are three subalgebras of S put

$$I_C(\mathfrak{A}, \mathfrak{B}|L) = \sup \sum_{i,j} \mu_x(A_i \wedge B_j) \log \frac{\mu_x(A_i \wedge B_j)}{\mu_x(A_i)\mu_x(B_j)},$$

where $x \in C$ and the sup is taken over all possible finite decompositions $A = \{A_i\}$, $B = \{B_j\}$ of X, $A_i \in \mathfrak{A}$, $B_j \in \mathfrak{B}$. Then $I_x(\mathfrak{A}, \mathfrak{B}|L) = I_C(\mathfrak{A}, \mathfrak{B}|L)$ for $x \in C$ is a measurable function on X. Let $MI(\mathfrak{A}, \mathfrak{B}|L) = \int_X I_x(\mathfrak{A}, \mathfrak{B}|L)d\mu$, $H_x(\mathfrak{A}|L) = I_x(\mathfrak{A}, \mathfrak{A}|L)$, $MH(\mathfrak{A}|L) = \int_X H_x(\mathfrak{A}|L)d\mu$. The above I_x, H_x, MI, MH are the conditional quantity of information, conditional mean information, conditional entropy and mean entropy of $\mathfrak{A}$ (w.r.t. given L). If $L = N$, $\mathfrak{A}$ = the trivial algebra, then $I_x = MI$, $H_x(\mathfrak{A}|N) = MH = H(\mathfrak{A})$ a.e., and $H(\mathfrak{A}) = \mathrm{Sup}\,[-\sum \mu(A_i) \log \mu(A_i)]$, where the sup is taken over all possible finite decompositions A of X. Several properties of the conditional information and entropy are stated. Some of these are generalizations of the properties of non-conditional information and entropy.

Let now $\{S_t\}$ denote a system given by an automorphism T, and let $\mathfrak{A}_0$ be a given closed subalgebra of S. Let $h(\mathfrak{A}_0) = \lim_{t-s}(t-s)^{-1}H(\mathfrak{A}_{s,t})$, which is stated to exist. For a simpler situation cf. Khinchin, *Mathematical foundations of information theory* [Dover, N.Y., 1957; MR **19**, 1148]. Let $h_1(T) = \inf_{\mathfrak{A}_0} h(\mathfrak{A}_0)$ where $\mathfrak{A}_0$ runs over all possible $\mathfrak{A}_0$'s satisfying $S(\mathfrak{A}_0) = S$. It is shown that $h(\mathfrak{A}_0)$ is the same for all such subalgebras $\mathfrak{A}_0$ with $h(\mathfrak{A}_0) < \infty$. Thus $h_1(T)$ is a metric invariant of the system.

Let $X = \prod_{-\infty}^{\infty} X_j$ where X_j for every j is the space of finite or denumerable number of points P_i each having the measure p_i. The automorphism T is taken to be the shift transformation $\{x_j\} \to \{x_j'\}$ where $x_j' = x_{j+1}$. This automorphism is quasi regular and hence has a Lebesgue spectrum of multiplicity $\aleph_0$. $h_1(T)$ can be computed to be $-\sum_i P_i \log P_i$. Thus for every $0 < h_1 \leq \infty$ there exists an automorphism having h_1 as its entropy and a Lebesgue spectrum of multiplicity $\aleph_0$. All these automorphisms are spectrally equivalent and metrically non-equivalent.

Y. N. Dowker (London)

2036a:

Sinaĭ, Ya. On the concept of entropy for a dynamic system. Dokl. Akad. Nauk SSSR **124** (1959), 768–771. (Russian)

2036b:

Sinaĭ, Ya. Flows with finite entropy. Dokl. Akad. Nauk SSSR **125** (1959), 1200–1202. (Russian)

Let T be an automorphism of a Lebesgue space $(\mathscr{X}, S, \mu)$. Let A be a finite measurable decomposition of $\mathscr{X}$; $\mathscr{X}=\bigcup_{i=1}^{n} A_i$, $A_i \in S$; $A_i \wedge A_j = \emptyset$ if $i=j$. The number

$$H(A) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i)$$

is called the entropy of A. Let T^kA denote the decomposition $\mathscr{X}=\bigcup_{i=1}^{n} T^kA_i$ and let $\bigvee_{j=m}^{p} T^jA$ denote the decomposition of $\mathscr{X}$ which is the common refinement of T^mA, $T^{m+1}A$, $\cdots$, T^pA. According to a well-known theorem [cf. Khinchin, op. cit., #2035a-b above]

$$\lim_{p\to\infty} p^{-1}H\left(\bigvee_{j=0}^{p} T^jA\right)$$

exists. Let this limit be denoted by $h(T, A)$. The author defines the entropy $h(T)$ of T by $h(T)=\operatorname{Sup}_A h(T, A)$ where A runs over all possible finite measurable decompositions of $\mathscr{X}$. (Notice that this definition differs somewhat from that of Kolmogorov's paper reviewed above). Let $S(A)$ denote the closed σ-algebra generated by sets of the form T^kA_i, $1 \leq i \leq n$, $-\infty < k < \infty$. Among other properties, the author proves that (1) $h(T^k)=|k|h(T)$ for every $-\infty < k < \infty$ and (2) if A is such that $S(A)=S$ then $h(T, A)=h(T)$. The author employs the last equality, judiciously choosing a decomposition, to find the entropy of an ergodic automorphism on a 2-dimensional torus. In this case $h(T)=\log|\lambda_1|$ where λ_1 and λ_2 ($|\lambda_1|=|1/\lambda_2|>0$) are the characteristic roots of the integral matrix with determinant ± 1 representing the automorphism. It is stated that the expected generalization to the n-dimensional torus holds.

Let $\{S_t\}$ denote a flow on $\mathscr{X}$ and let Δ be a real number. Then S_Δ is an automorphism of $\mathscr{X}$. The author defines the entropy $h(\{S_t\})$ of the flow $\{S_t\}$ by

$$h(\{S_t\}) = \operatorname*{Sup}_{\Delta>0} \frac{h(S_\Delta)}{\Delta}.$$

With this definition the entropy is an invariant of the flow and is used by the author to distinguish between flows with the same continuous $\aleph_0$-multiple Lebesgue spectrum. The two kinds of flows in question have both originated with Kolmogorov. [The second and more complicated one appeared in the first paper of Kolmogorov reviewed above.] The first example is described as follows: $\{S_t\}$ is defined by a stationary Markov process whose phase space is made up of two segments A_1B_1 and A_2B_2 of lengths α and β respectively. On each segment there is a motion to the right of unit speed. From the ends B_1 and B_2 of the segments there is a jump to A_1 or A_2 with equal probability $\frac{1}{2}$. The distribution on each segment is taken to be uniform. The entropy of $\{S_t\}$ in this case is computed to be $2/(\alpha+\beta)$. If α and β are non-commensurable the spectrum is stated to be $\aleph_0$-multiple Lebesgue. Thus we obtain for every real number $0<h\leq\infty$ a flow having h as its entropy and a Lebesgue spectrum of multiplicity $\aleph_0$. All these flows are spectrally isomorphic but metrically non-isomorphic. *Y. N. Dowker* (London)

2037:

Rohlin, V. A. Entropy of metric automorphism. Dokl. Akad. Nauk SSSR **124** (1959), 980–983. (Russian)

Let (X, S, μ) be a Lebesgue space with a continuous measure. Let A be a finite or denumerable decomposition of X into measurable sets A_i of measure μ_i. The number $H(A)=-\sum_i \mu_i \log \mu_i$ is called the entropy of A. Let $Z=\{A \mid H(A)<\infty\}$. Let $\bigwedge A_\alpha$ be the common refinement of the decompositions A_α. If $A \in Z$, $B \in Z$, the conditional entropy of A given B is defined along the lines of Kolmogorov [cf. #2035a-b above] or by extension from the finite case as $H(A|B)=\sum \mu_i H_i(B)$ where

$$H_i = \sum_k \frac{\mu(A_i \wedge B_k)}{\mu(A_i)} \log \frac{\mu(A_i \wedge B_k)}{\mu(A_i)}$$

[cf. Khinchin]. Let $\rho(A, B)=H(A|B)+H(B|A)$. Then Z is a complete separable metric space with respect to ρ with the set of finite decompositions being everywhere dense in Z. Let $h(T, A)=\lim_{n\to\infty} H(\bigwedge_{k=0}^{n-1} T^kA)$. This limit exists, according to a well-known theorem [cf. Khinchin, cited above]. It is shown that $h(T, A)$ is a continuous function of A on Z. Let $h(T)=\operatorname{Sup}_{A\in Z} h(T, A)$. $h(T)$ is called the entropy of T. It is stated that the definition is equivalent to that of Sinaĭ and Kolmogorov [cf. reviews above]. The reviewer however was unable to see the equivalence of the Sinaĭ-Rohlin definition with that of Kolmogorov without some restrictive conditions. Let $\mathfrak{N}_s$ and $\mathfrak{N}_w$ be the space of automorphism of X with the uniform and weak topologies respectively. [Cf. Halmos, *Lectures on ergodic theory*, Math. Soc. Japan, 1956; MR **20** #3958.] Then the set of all automorphisms is a dense G_δ in $\mathfrak{N}_s$ and $\mathfrak{N}_w$. Some relation between the entropy of T and its spectrum is given by the following theorem: If $h(T)>0$ then $L_2(\mu)$ contains an invariant subspace on which the spectrum is $\aleph_0$-multiple Lebesgue. Thus, for instance, if T has a pure point spectrum, $h(T)=0$. A is called a generator for T if $\bigwedge_{k=-\infty}^{\infty} T^kA=\varepsilon$ (mod 0). If $A \in Z$ is a generator then $h(T)=h(T, A)$. The problem as to whether every ergodic transformation has a generator from Z is unsolved. Some results as to the size of sets with simple generators from Z as well as flows are also given. *Y. N. Dowker* (London)

2038:

Cafiero, Federico. Sul principio esteso delle probabilità totali. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 85–102 (1957).

Let $\mathscr{R}$ be a Boolean ring of subsets (that we may regard as "events") of a set $\mathscr{S}$, and $\mathscr{C} \subset \mathscr{R}$ a "σ-chain" of subsets, i.e. a class of subsets such that if $E_1 \in \mathscr{C}$, $E_2 \in \mathscr{C}$, then either $E_1 \subset E_2$ or $E_2 \subset E_1$, and if $E_k \in \mathscr{C}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathscr{C}$. Let $\mathscr{C}$ have at least the power of the continuum, and suppose that $\bigcap_{k=1}^{\infty} E_k \in \mathscr{C}$, where $E_k \in \mathscr{C}$, if and only if $\bigcap_{k=1}^{\infty} E_k$ is a set of the sequence (E_k); then, if ψ is a finitely additive measure (for instance a probability function) on $\mathscr{R}$, there exists a σ-ring $\mathscr{R}' \subset \mathscr{R}$, having at least the power of the continuum, such that ψ is countably additive on $\mathscr{R}'$. The theorem is proved by showing that there exists a sub-chain of $\mathscr{C}$ in all subsets of which the function ψ is continuous from above. *D. Fürst* (Rome)

2039:

Turner, L. H. An invariant property of Cesari's surface integral. Proc. Amer. Math. Soc. **9** (1958), 920–925.

The author gives a modified definition for Cesari's surface integral and shows that it is equivalent to the original one [L. Cesari, *Surface area*, Princeton Univ. Press,

Princeton, N.J., 1956; MR **17**, 596; Appendix B]. The new definition is in terms of the vector-valued set-function $\mathscr{V}=(\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3)$, where $\mathscr{V}_r = V_r^+ - V_r^-$, V_r^+ and V_r^- being defined, as by Cesari [loc. cit.], as the signed variations of partial plane transformations, $r=1, 2, 3$. Then the coordinates of $\mathscr{V}$ change covariantly under rotations of 3-space, from which Cesari's surface integral is invariant under such rotations.

W. H. Fleming (Providence, R.I.)

2040:

Fleming, W. H. Nondegenerate surfaces and fine-cyclic surfaces. Duke Math. J. **26** (1959), 137–146.

Let T be a continuous mapping from a compact finitely connected Jordan region J in the plane into n-space. The mapping T is said to be fine-cyclic if T is not constant on J and no set which is the union of a finite number of maximal continua of constancy for T disconnects J. If T is fine cyclic and the Lebesgue area of the surface S represented by T is finite then it is proved that the Lebesgue area is equal to the Geöcze area; moreover, it is shown that there exists a representation T_0 for S defined on a compact finitely connected Jordan region J_0 in the plane, and a compact set K in J_0, with the following properties: J_0 is bounded by circles; K is the union of finitely many analytic arcs; J_0-K is connected; T_0 is constant on each component of K; on the interior of any Jordan region which is contained in J_0-K the components of T_0 are absolutely continuous in the sense of Tonelli and the function E_0+G_0 is Lebesgue summable; the Lebesgue area of S is given by the Lebesgue integral of $(E_0G_0-F_0^2)^{1/2}$ over the interior of J_0.

P. V. Reichelderfer (Columbus, Ohio)

2041:

Cesari, L.; and Neugebauer, Ch. J. On the coincidence of Geöcze and Lebesgue areas. Duke Math. J. **26** (1959), 147–153.

Let (T, A) be a continuous mapping from an admissible set A of the Euclidean plane into Euclidean three-space [L. Cesari, *Surface area*, Princeton Univ., Princeton, N.J., 1956; MR **17**, 596]. Using their concepts of fine-cyclic elements [Cesari, Riv. Mat. Univ. Parma **7** (1956), 149–185; MR **19**, 1168; Neugebauer, Trans. Amer. Math. Soc. **88** (1958), 121–136; MR **20** #1959] and fine-cyclic additivity theorems, the authors give a new proof for the known result that the Lebesgue area and the Geöcze area of (T, A) are equal. *P. V. Reichelderfer* (Columbus, Ohio)

FUNCTIONS OF A COMPLEX VARIABLE

See also 1953, 1977, 2064, 2091, 2120, 2150, 2550.

2042:

Andreian Cazacu, Cabiria. Soviet achievements in the theory of pseudo-analytic functions. Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **13** (1959), no. 2 (29), 69–76. (Romanian. Russian summary)

Brief survey article.

2043:

Tomić, M. Sur les polynômes de Fejér. Glas Srpske Akad. Nauka **232** Od. Prirod.-Mat. Nauka (N.S.) **15** (1958), 29–44. (Serbo-Croatian. French summary)

Serbo-Croatian version of the paper reviewed below.

2044:

Tomić, M. Sur les polynômes de Fejér. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 39–52.

The author considers polynomials of the form

$$p_n(z) \equiv \sum_1^n a_k(z^{n-k} - z^{n+k-1})$$

with $a_k \downarrow 0$. A special case, corresponding to $a_k = 1/k$, is the Fejér polynomials, whose principal property is that they are uniformly bounded on the unit circle C. Certain other results, concerning the maxima of the Fejér polynomials on C and the position of their zeros, were proved by G. Piranian and the reviewer [Proc. Amer. Math. Soc. **7** (1956), 379–386; MR **18**, 26]. In the present paper the author obtains similar results for the more general case of the polynomials $p_n(z)$. He proves a condition for their uniform boundedness on C, namely, $ka_k < M$. Under the assumption that the a_k satisfy certain conditions in addition to $a_k \downarrow 0$, he obtains estimates for the maximum of $|p_n(z)|$ on C, the distribution of the zeros of $p_n(z)$ with respect to their argument and, in the case of even n, an asymptotic formula for the two negative roots of $p_n(z)$. Finally, the trigonometric polynomials

$$T_n(\theta) \equiv \tfrac{1}{2} + \sum_1^n (\cos k\theta)/(k+1), \quad R_n(\theta) \equiv 1 + \sum_1^n (\cos k\theta)/k,$$

known to be non-negative for real θ, are shown to have the lower bounds 1/168 and 1/20, respectively, for $n \geqq 2$.

{The paper contains a large number of errors, more or less serious, which make parts of it very hard to follow. In theorem IV some additional assumption, such as $\rho(n) = O(n^{1/2})$, seems necessary in order to make the proof valid for the root which is less than -1. The result concerning $T_n(\theta)$ is based on the obviously erroneous inequality $(\sin t)/t \geqq 7/2\pi$ for $0 < t < \pi/6$. And the exact value of the minimum of $R_n(\theta)$, taken over all real θ and $n \geqq 2$, is $R_3(\pi) = 1/6$.} *F. Herzog* (East Lansing, Mich.)

2045:

Kuipers, L.; and Veldkamp, G. R. Commentary on a special case of the Gauss-Lucas theorem in the geometry of the zeros of polynomials. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 430–433.

This paper contains another simple proof of the oft-proved theorem that for a polynomial

$$f(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}(z-z_3)^{m_3}$$

the derivative $f'(z)$ has its non-multiple zeros at the foci of a conic which is tangent to the sides of the triangle joining the points z_1, z_2, and z_3, and that the points of tangency divide the sides of this triangle in the ratios m_1/m_2; m_2/m_3; m_3/m_1. [For references to other proofs see M. Marden, *Geometry of the zeros of a polynomial in a complex variable*, Amer. Math. Soc., New York, 1949; MR **11**, 101.] *M. Marden* (Milwaukee, Wis.)

2046a:

Shah, Tao-shing. Some covering properties of convex

domains in the theory of conformal mapping. Acta Math. Sinica **7** (1957), 421–432. (Chinese. English summary)

2046b:

Shah, Tao-shing. Some covering properties of convex domains in the theory of conformal mapping. Sci. Sinica **7** (1958), 816–828.

[The second paper is a translation of the first.]

A convex domain G in the w-plane has a supporting radius ρ if through each point of the boundary of G passes a circle of radius ρ ($\rho>0$) containing G in its interior, but there is some point at which no smaller circle does this. Let K_ρ denote those convex domains of supporting radius ρ which are images of $|z|<1$ under functions of the class C_ρ normalized by $f(z)=z+a_2z^2+\cdots$. Let r_k ($k=1, 2, \cdots, n$) represent n rays making equal angles at the origin and let γ_k denote the length of the segment r_k which is in G ($G\in K_\rho$). The determination of

$$T_n(\rho)=\min_{G\in K_\rho}\max\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$$

is called the Szegö problem for the class C_ρ. The author proves that

$$T_n(\rho)=\rho\left(1-\frac{\sin x\pi}{\sin(\pi/n)}\right)\quad (1<\rho<\infty;\ n>1),$$

where x is the root of

$$(1-xn)\rho\Gamma\left(\frac{n-1}{2}\right)^2=\Gamma\left(\frac{n-1}{n}-x\right)\Gamma\left(\frac{n+1}{n}+x\right)$$

satisfying $0<x<1/n$. Also $T_1(\rho)=\rho-(\rho^2-\rho)^{1/2}$ ($1\leqq\rho<\infty$) with the extremal function $f(z)=z/[1+(1-\varepsilon z/\rho)^{1/2}]$, $|\varepsilon|=1$. Furthermore $T_n=\lim_{\rho\to\infty}T_n(\rho)=\int_0^1(1+t^n)^{-2/n}dt$. He uses these results to show that a Bloch function for C_ρ maps $|z|<1$ onto a region bounded by two circular arcs each of radius ρ and symmetric in $w=0$. These results are proved using the method of extremal length.

G. Springer (Lawrence, Kansas)

2047:

Jurchescu, Martin. Modulus of a boundary component. Pacific J. Math. **8** (1958), 791–809.

It is known that the capacity c_γ of a Kerékjártó-Stoïlow component γ of the ideal boundary β of a Riemann surface R can be defined in a conformally invariant manner as $c_\gamma=e^{-k_\gamma}$ with $k_\gamma=\min(1/2\pi)\int_\beta t\,dt^*$. Here t ranges over the class of harmonic functions on R with a fixed singularity $\log|z-\zeta|$, with conjugate periods 2π along cycles separating γ from ζ, and with periods 0 along all other cycles [L. Sario, Ann. of Math. (2) **59** (1954), 135–144; MR **15**, 518]. By definition, γ is weak if $c_\gamma=0$; β is absolutely disconnected if every γ is weak; R belongs to the class C_γ if β is absolutely disconnected.

In the present paper conditions are derived for γ to be weak and for R to belong to C_γ. The above boundary integral is taken for functions harmonic in a neighborhood $S\subset R$ of γ. The normalizing condition is $u=0$ on the relative boundary γ_0 of S, and the conjugate periods are, mutatis mutandis, the same as for t. The minimum of the integral is defined as the modulus μ_γ of γ. It is shown that γ is weak if and only if $\mu_\gamma=\infty$. The minimizing function u_γ also furnishes the metric $\rho_\gamma=|\text{grad } u_\gamma|$ with the smallest area integral $\int_S\rho^2d\sigma$ among conformal metrics ρ that satisfy the condition $\int_c\rho|dz|\geqq1$ for cycles c separating γ from γ_0.

Another necessary and sufficient condition for the weakness of γ is given: for every admissible ρ there exists a c separating γ from γ_0 with an arbitrarily small $\int_c\rho|dz|$. Moreover, γ is weak if and only if there exists an exhaustion $\{S_n\}$ of S with the property that $\sum\mu_n$ diverges for suitably normalized moduli μ_n of the components of S_n-S_{n-1} that tend to γ. This is the counterpart of N. Savage's sufficient condition for $c_\gamma=0$ [Duke Math. J. **24** (1957), 79–95; MR **18**, 647]. The author establishes the equivalence of c_γ and the Ahlfors-Beurling perimeter

$$\pi_\gamma=\lim_{r\to0}(1/r)\exp(-2\pi\mu_{\gamma r}),$$

where $\mu_{\gamma r}$ is the modulus of γ with respect to the complement of a disk $|z|<r$ [Acta Math. **83** (1950), 101–129; MR **12**, 171].

Connections with the extendability of R are studied. For the case of a finite genus, the following three conditions are shown to be equivalent: $R\in C_\gamma$; the continuation of R is "topologically unique"; R does not have essential continuations.

L. Sario (Los Angeles, Calif.)

2048:

Valiron, Georges. Fonctions entières d'ordre fini et fonctions méromorphes. I, II, III, IV. Enseignement Math. (2) **4** (1958), 1–18, 124–156, 157–177, 229–271.

This is a posthumously published series of lectures on various aspects of the theory of entire functions, drawn chiefly from the topics that have been the chief interests of its author. There is much in common with his previous books on the same subject, and with the more recently published treatises by Boas and by Cartwright. Since the latter cover a wider field, with more attention to contemporary material, and contain more detailed proofs, a student would be less likely to find these papers by Valiron of use to him. However, they are written with skill and economy, well suited to their intended audience.

R. C. Buck (Princeton, N.J.)

2049:

Srivastav, R. P. A note on the maximum term of an integral function. Ganita **8** (1957), 23.

A short proof is given for the known result

$$\log\mu(r)=\log\mu(r_0)+\int_{r_0}^r t^{-1}\nu(t)dt\quad (r=|z|),$$

where $\mu(r)$ and $\nu(r)$ are the maximum term and the rank of the maximum term, respectively, of the expansion $\sum a_nz^n$ of an entire function.

H. Kober (Birmingham)

2050:

Rényi, Catherine. On periodic entire functions. II. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 123–126.

[For Part I see Acta Math. Acad. Sci. Hungar. **8** (1957), 227–233; MR **19**, 539.] Let $Z_x(n)$, $C_x(n)$, $P_x(n)$, $B_x(n)$ denote the number of zeros, the number of changes of sign, the number of permanences of sign, and the number of blocks of zeros in the sequence $f(x), f'(x), \cdots, f^{(n)}(x)$; and let $S_{\alpha,\beta}(n)=Z_\alpha(n)+P_\alpha(n)+Z_\beta(n)+C_\beta(n)-n$. The author sharpens some results of part I by proving that if

f is an entire function of finite order $\rho \geqq 1$, real on the real axis, and if α and β are real and $\beta - \alpha = 2R > 0$, then

$$\liminf S_{\alpha,\beta}(n) n^{-1+1/(\rho+\epsilon)} \leqq 0$$

for any $\epsilon > 0$. If, further, f is of finite type τ, then

$$\liminf S_{\alpha,\beta}(n) n^{-1+1/\rho} \leqq Re^2(\tau/e)^{1/\rho}.$$

Now put $D_x(n) = Z_x(n) - B_x(n)$ and suppose that f is, in addition to the previous hypotheses, periodic with period P. Then

$$\liminf D_x(n) n^{-1+1/(\rho+\epsilon)} \leqq 0,$$

and if f is of finite type τ

$$\liminf D_x(n) n^{-1+1/\rho} \leqq \tfrac{1}{2}|P|e^2(\tau/e)^{1/\rho}.$$

For $\rho = 1$ this says that for a nonconstant entire function of exponential type τ and period P the number of derivatives having multiple roots at a given point does not exceed $\frac{1}{2}|P|e\tau$; but since a periodic entire function of exponential type is necessarily a trigonometric polynomial, the constant $e/2$ can be sharpened, as the author will show in a forthcoming paper.

R. P. Boas, Jr. (Evanston, Ill.)

2051:

Tsuji, Masatsugu. Canonical product for a meromorphic function in a unit circle. J. Math. Soc. Japan 8 (1956), 7–21.

Let $w(z)$ be a meromorphic function in $|z| < 1$ of finite order ρ with zeros a_n $(n = 1, 2, \cdots)$ in $|z| < 1$, $a_n \neq 0$. The canonical product is defined as follows. If

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty,$$

then $P(z) = \prod_{n=1}^{\infty} (1 - A_n(z))$, where

$$A_n(z) = (1 - |a_n|^2)/(1 - \overline{a_n} z).$$

If $\sum_{n=1}^{\infty} (1 - |a_n|) = \infty$, the exponent of convergence of the a_n is defined as the positive number μ such that $\sum_{n=1}^{\infty} (1 - |a_n|)^{\mu+1-\epsilon} = \infty$ and $\sum_{n=1}^{\infty} (1 - |a_n|)^{\mu+1+\epsilon} < \infty$ for any $\epsilon > 0$. Let $p \geqq 1$ be a positive integer such that $\sum_{n=1}^{\infty} (1 - |a_n|)^p = \infty$, and $\sum_{n=1}^{\infty} (1 - |a_n|)^{p+1} < \infty$. Then

$$P(z) = \prod_{n=1}^{\infty} (1 - A_n(z)) \exp\left[A_n(z) + \tfrac{1}{2}A_n(z)^2 + \cdots + p^{-1}A_n(z)^p\right].$$

The author proves several theorems relating the order ρ^* of $P(z)$, μ, p, and ρ. For example, if $\sum_{n=1}^{\infty} (1 - |a_n|) = \infty$, $\rho^* = \mu \leqq \rho$. From this he concludes that every meromorphic function of finite order ρ in $|z| < 1$ can be expressed as a ratio of two regular functions of order $\leqq \rho$ in $|z| < 1$. He also proves the theorem that if $w(z)$ is meromorphic and of order $\rho (\leqq \infty)$ in $|z| < 1$, and ρ' is the order of $w'(z)$, then $\rho' = \rho$.

G. Springer (Lawrence, Kans.)

2052:

Ogawa, Shôtaro; and Sakaguchi, Kôichi. Some classes of meromorphic functions with assigned zeros and poles. J. Math. Soc. Japan 8 (1956), 40–53.

Let $f(z)$ be meromorphic in $|z| < 1$ and $f(z) \neq 0, \infty$ for $|z| = 1$. If there are two rays starting at the origin which are crossed $2p$ times by $f(e^{i\theta})$ as θ varies from 0 to 2π, then $f(z)$ is said to be of order p in the directions of the two rays. If the minor angle between the two rays is α, then f belongs to the class $M(p, \alpha)$. A point $f(e^{i\theta_\nu})$ on either of the two rays is called a positively or negatively cutting point depending upon whether the angular velocity about the origin of $f(e^{i\theta})$ is positive or negative as $e^{i\theta}$ traverses $|z| = 1$ in the positive direction. The general theorem proved by the authors is the following. Let $f \in M(p, \lambda\pi)$ and $f(z) = z^q + \sum_{n=q+1}^{\infty} c_n z^n$, where q is an integer which may be positive, negative, or zero. In $0 < |z| < 1$, let f have exactly l zeros, $a_1, a_2, \cdots, a_l$, and m poles, $b_1, b_2, \cdots, b_m$. Let $f(e^{i\theta_\mu})$, $\mu = 1, 2, \cdots, 2p - t$, and $f(e^{i\theta_\nu})$, $\nu = 1, 2, \cdots, t$, denote the positively and negatively cutting points respectively, where $t = p - (q + l - m)$. Then for $0 \leqq r < 1$, one has

$$r^q\left(\frac{1-r}{1+r}\right)^{2-\lambda} \min_{|z|=r} |A(z)B(z)| \leqq |f(re^{i\theta})| \leqq r^q\left(\frac{1+r}{1-r}\right)^{2-\lambda} \max_{|z|=r} |A(z)B(z)|,$$

and

$$\{z^{-q}f(z)/A(z)B(z)\}^{1/(2-\lambda)} \ll \frac{1+z}{1-z},$$

where

$$A(z) = \prod_{j=1}^{l} (1 - \bar{a}_j z)\left(1 - \frac{z}{a_j}\right) \Big/ \prod_{k=1}^{m} (1 - \bar{b}_k z)\left(1 - \frac{z}{b_k}\right),$$

and

$$B(z) = \prod_{\nu=1}^{t} (e^{i\theta_\nu} - z) \Big/ \prod_{\mu=1}^{2p-t} (e^{i\theta_\mu} - z),$$

and the symbol $\ll$ means that the moduli of the coefficients in the power series expansion of the left side are dominated by those of the right side. Special cases of this theorem for $M(p, \pi)$ and $M(p, 0)$ are given in a series of corollaries.

G. Springer (Lawrence, Kans.)

2053:

Wittich, H. Defekte Werte eindeutiger analytischer Funktionen. Arch. Math. 9 (1958), 65–74.

In the first part of the paper, results are established on the deficient values of the derivative of a meromorphic function $w(z)$ with some maximal deficiency property. Direct use is hereby made of the main inequality of Nevanlinna concerning the characteristic function of $w'(z)$. In standard notation, a typical result is as follows: If $w(z)$ is meromorphic and non-rational for $z \neq \infty$ and $\sum \delta(w, a) = 2$, $a \neq \infty$, then $\lim T(r, w')/T(r, w) = 2$, $\delta(w', 0) = 1$, $\delta(w', \infty) = 0$, $\sum \delta(w', b) \leqq \frac{1}{2}$, $b \neq 0, \infty$.

In the latter part, the interesting relation

$$\lim N(r, c)/T(r, w) = 1$$

is established for every meromorphic $w(z)$ of unbounded characteristic and for all values c, except for a set of interior capacity zero. The result is applied to a problem in the theory of ordinary differential equations.

O. Lehto (Helsinki)

2054:

Gehring, F. W. The asymptotic values for analytic functions with bounded characteristic. Quart. J. Math. Oxford Ser. (2) 9 (1958), 282–289.

Let $f(z)$ be regular in $|z| < 1$ and P be a point on $|z| = 1$; let $\Gamma(P, f)$ be the set of the asymptotic values of $f(z)$ at P and $\Gamma_\Delta(P, f)$ be the same under the condition that the

asymptotic path is contained in a Stolz domain whose vertex is at P. $f \in N$ denotes that $\log^+|f|$ has a finite harmonic majorant in $|z|<1$. Let $\phi(t)$ be a non-negative convex function, such that $\phi(t)/t \to \infty$ $(t \to \infty)$. If $\phi(\log^+|f|)$ has a finite harmonic majorant in $|z|<1$, then we write $f \in N^*$. If $f \in N$, then $f(z)$ is of bounded characteristic. The author proves first lemma 1: $f \in N^*$ when and only when $\log^+|f|$ has a finite harmonic majorant in $|z|<1$ which can be written as a Poisson integral; and then theorem 1: Let $f \in N$; then for each P on $|z|=1$, $\Gamma(P, f)$ contains at most two finite values. If $\Gamma_\Delta(P, f)$ contains a finite value, then $\Gamma(P, f)$ contains only one finite value. Theorem 2: Let $f \in N^*$; then for each P on $|z|=1$, $\Gamma(P, f)$ contains at most one finite value. Theorem 3: Let $f \in N$ and $f \neq 0$ in $|z|<1$; then for each P on $|z|=1$, $\Gamma(P, f)$ contains at most one finite value. If $\Gamma(P, f)$ contains a $(\neq 0, \infty)$, then $\Gamma(P, f)$ contains only a. At the end of the note, the author constructs a function, which is meromorphic and of bounded characteristic in $|z|<1$, such that $\Gamma_\Delta(1, f)$ has the power of the continuum. *M. Tsuji* (Tokyo)

2055:

Suvorov, G. D. Boundary correspondence in topological mapping of plane regions with variable boundaries. Dokl. Akad. Nauk SSSR **124** (1959), 772–774. (Russian)

The author applies the theory of prime ends of a nuclear sequence of domains [Suvorov, Mat. Sb. (N.S.) **33** (**75**) (1953), 73–100; MR **15**, 244] to further extensions of Caratheodory's conformal representation theorems. The extensions concern certain sequences of one to one mappings of a disc onto plane domains, the Dirichlet integrals being supposed uniformly bounded.

L. C. Young (Madison, Wis.)

2056:

Vituškin, A. G. Analytic capacity of sets and some of its properties. Dokl. Akad. Nauk SSSR **123** (1958), 778–781. (Russian)

Let E be a bounded point set in the z-plane. Denote by $A(E)$ the set of all functions $\varphi(z)$ regular in the complement CE of E and satisfying $|\varphi(z)| \leq 1$, $\varphi(\infty)=0$ there. The analytic capacity $\gamma(E)$ of E is defined as $\sup \lim_{z\to\infty} |z\varphi(z)|$ as φ varies over all of $A(E)$.

The point z is called a density-point of E, if

$$\sum_{n=1}^{\infty} 4^n \gamma(CE \cap K_n) < \infty,$$

where K_n is the annulus $\{\zeta | 2^{-n-1} \leq |\zeta - z| \leq 2^{-n}\}$.

Both these definitions are motivated by an important application to the theory of approximation [see #2057 below]. In this note the author develops properties of the analytic capacity. Many of these are analogous to the properties of the ordinary (harmonic) capacity. Two typical results are: (1) If Γ is the rectifiable boundary of a domain G and if $E \subset G$ has distance $\geq \delta > 0$ from Γ, then for every function regular in $G-E$

$$\left| \int_\Gamma f(z)dz \right| \leq K(G, \delta) \sup_{G-E} |f(z)| \gamma(E);$$

(2) if the set E has no density-points, then for every circle K: $|z-a|<\rho$, $\gamma(CE \cap K) \geq A\rho$ (A an absolute constant which is actually equal to 1 by a result of the paper reviewed below). *W. H. J. Fuchs* (Ithaca, N.Y.)

2057:

Vituškin, A. G. Some theorems on the possibility of a uniform approximation of continuous functions by analytic functions. Dokl. Akad. Nauk SSSR **123** (1958), 959–962. (Russian)

In this paper the notions developed in the note reviewed above are applied to the theory of approximation of continuous functions by analytic functions. The results come very close to giving necessary and sufficient conditions for the possibility of such approximation. Typical theorems follow.

Theorem 1: If E has no density points, then every function defined and continuous on E can be approximated uniformly with arbitrary closeness by a function regular in some neighborhood of E.

Theorem 2: If E is closed and if there is an $\varepsilon > 0$ such that, for every boundary point z of E, $\gamma(CE \cap K) \geq \varepsilon\rho$ where K is an open circle of center z and radius $\rho \leq \rho_0(z)$, then a necessary and sufficient condition for the uniform approximability of $f(z)$ on E by rational functions is that $f(z)$ is continuous on E and has a derivative at every interior point of E.

Theorem 3: Let $\kappa(E)$ be the (harmonic) capacity of E. If every complex-valued function on E which is uniformly continuous on E can be uniformly approximated by functions regular on E, then $\kappa(CE \cap K) \geq Ar$ where K is a circle of radius r and A is an absolute constant.

The proofs follow the lines of Mergelyan's investigation of approximation by analytic functions [S. N. Mergelyan, Uspehi Mat. Nauk **7** (1952), No. 2 (48), 31–122; MR **14**, 547]. The new element in the proof is an approximation of $g(\zeta) = 1/(\zeta - z)$ $(z \in E)$ by a function regular in CE and bounded by a constant depending on $\gamma(E)$.

W. H. J. Fuchs (Ithaca, N.Y.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

2058:

★Hirzebruch, Friedrich. Automorphe Formen und der Satz von Riemann-Roch. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 129–144. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

Let X be a bounded symmetric domain of the complex number space C^n. Corresponding to X there is a naturally associated compact homogeneous algebraic variety X', into which X can be imbedded as an open submanifold in such a manner that all analytic automorphisms of X can be extended to analytic automorphisms of X'. Suppose that Δ is a discontinuous group of automorphisms of X without fixed points, so that X/Δ is compact. Then X/Δ is known to be a projective variety. In this paper it is shown how problems such as the existence of automorphic forms of X/Δ can be brought into relation with corresponding problems over X'. In this latter situation one has explicit information due to earlier work of Borel and Hirzebruch [Amer. J. Math. **80** (1958), 458–538; MR **21** #1586] which describes the cohomology of X' with coefficients in homogeneous vector bundles in terms of the representations which induce them. The link between X and X' is determined by the curvature tensors of these

two varieties, which are negatives of each other. Using the curvature definition of the Chern classes it then follows easily that X/Δ and X' are proportional, in the sense that every Chern number of X/Δ is a fixed multiple of the corresponding Chern number of X'. As the author shows, the proportionality factor is precisely the arithmetic genus of (X/Δ). Thus $c_\nu(X/\Delta)=\chi(X/\Delta)\cdot c_\nu(X')$. The irreducible X have been classified by E. Cartan. Using this information and the theory developed here the author brings explicit formulas for the number $\Pi_r(X,\Delta)$, of automorphic forms of weight $r\geq 2$, on X/Δ in several cases. For instance, for the domain X whose compact associate X' is given by $U(p+q)/U(p)\times U(q)$, he obtains the following formula:

$$\Pi_r(X,\Delta) = (-1)^{pq}\chi(X/\Delta)\prod_{i,j}\frac{r(p+q)-i-j}{p+q-i-j}$$

$$(0\leq i\leq p-1,\ 1\leq j\leq q).$$

R. Bott (Ann Arbor, Mich.)

2059:

Iwahashi, Ryōsuke. Domains spread on a complex space. J. Math. Soc. Japan **9** (1957), 452–463.

This paper generalizes results [vid. *Séminaire H. Cartan* 1951–52, mimeographed; MR **16**, 233; Exp. VII, Th. 1 and Th. 1 bis] on the existence of maximal prolongations of spread domains from the case of connected analytic manifolds to that of complex spaces in the sense of Behnke-Stein-Cartan [vid. e.g. MR **17**, 80–83].

Let X, E, and F be complex spaces; then the space of holomorphic jets $A(E,F)=\bigcup A_x$, where A_x is the inductive limit in the obvious sense of the sets A_U of holomorphic maps of a non-empty open subset U of E into F; $\Gamma(U, A(E,F))$ is the set of sections over U, defined as for sheaves. Let f be a holomorphic map of X into F; (X,f) is a domain spread on F if f is open and holomorphic, and non-degenerate at each x in X in the sense that there exists a neighborhood U of x such that $f^{-1}(f(x))\cap U$ is discrete in U. Majorization of f and holomorphic prolongation of h in $\Gamma(X, A(X,F))$ are defined as in *Séminaire Cartan*, op. cit. The author proves the following theorems: 1. Given X, f, E, h as above, and connected, there exists among those f' which spread X and are majorized by f, and with respect to which h is holomorphically prolongable, an element majorizing all the others. 2. A similar result is proved for families of functions. The proofs use various results of Grauert and Remmert, and the construction of a space of 'algebroidal jets' from the analytic subsets of open $U\subset E$.

F. D. Quigley (New Orleans, La.)

2060:

Bochner, S. Linear and algebraic dependence of functions on compact complex spaces with singularities. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 47–49.

L'auteur énonce sans démonstration les résultats suivants. Soit U une variété analytique complexe, réunion finie de boules disjointes de dimension complexe n. Soit T un ouvert relativement compact de U, et soit Φ une application $U\to T$. On considère les fonctions f holomorphes dans U, à valeurs dans C^n, qui satisfont à (1) $f(z)=\lambda(z)f(\Phi(z))$, où $\lambda(z)$ est une $(N\times N)$-matrice bornée quand z varie. Ces f sont localement constantes si $\lambda(z)\equiv 1$, et forment un espace vectoriel de dimension finie dans le cas général. Le quotient de deux fonctions holomorphes scalaires satisfaisant à (1) (donc pour $N=1$) est méromorphe et invariante par Φ; $n+1$ telles fonctions méromorphes sont algébriquement dépendantes. L'auteur signale la possibilité de généraliser ces résultats au cas où U est un espace analytique avec singularités, mais ne donne pas de formulation précise.

{Erratum: page 48, ligne 11, il semble qu'on doive lire U_a au lieu de V_a.} *H. Cartan* (Paris)

2061:

Stoll, Wilhelm. Über meromorphe Abbildungen komplexer Räume. I. Math. Ann. **136** (1958), 201–239.

On considère essentiellement une application holomorphe τ d'un ouvert A d'un espace complexe G dans un espace complexe H, $G-A=M$ étant soumis à la restriction d'être "mince"; on dit que M est mince (respectivement mince d'ordre q) s'il est contenu au voisinage de chaque point dans un sous-ensemble analytique complexe (respectivement dans un tel sous-ensemble de codimension q au moins); M est dit presque mince s'il est contenu dans une réunion dénombrable d'ensembles minces; les ensembles considérés ainsi sont encore des ensembles polaires complexes [cf. P. Lelong, J. Math. Pures Appl. **36** (1957), 263–303; MR **19**, 1194]. L'application τ est dite sans lacune quand $\tau(P_\nu)$ renferme une sous-suite convergente, pour toute suite $P_\nu\to P_0\in M$, $P_\nu\in A$; elle est dite R-méromorphe quand son graphe T a une adhérence $\overline{T}$ dans $G\times H$ qui est un ensemble analytique [cf. R. Remmert, Math. Ann. **133** (1957), 328–370; MR **19**, 1193]. L'auteur introduit d'autres notions, moins restrictives que la définition précédente (R-méromorphie) de Remmert; en considérant les ensembles $\Sigma_\tau(P,L)$ des valeurs limites de τ quand on tend dans A vers un point $P\in M$ en demeurant sur une variété L à une dimension complexe: τ est dite méromorphe [respectivement faiblement méromorphe] dans G quand $\Sigma_\tau(P,L)$ consiste toujours en un point [respectivement en un point au plus]. Le problème posé est alors de préciser sous quelles conditions une application d'un de ces types est R-méromorphe. Il est lié, ainsi qu'on le voit en considérant le graphe de τ, à l'étude du prolongement d'un ensemble analytique dans $A\times H$ à tout l'espace $G\times H$, à travers l'ensemble mince $M\times H$. Les résultats obtenus peuvent être considérés comme une approche de ce difficile problème.

Un ensemble analytique $N\subset A$ est dit séparé de $M=G-A$ si à tout point $P\in M$ correspond un voisinage U et un ensemble analytique $M_0\supset[M\cap U]$, M_0 ne contenant aucune partie irréductible de l'ensemble analytique $N\cap U$ dans $U\cap A$. Pour que $\bar{N}$ prolonge N comme ensemble analytique dans G, il faut et il suffit que N soit séparé de $M=G-A$; le graphe T de τ est alors un ensemble analytique dans $A\times H$ qui est séparé de $M\times H$; si T est prolongeable dans $G\times H$, son prolongement est $\overline{T}$, mais $\overline{T}$ n'est pas nécessairement irréductible localement au-dessus d'un point $P\in M$, comme le montre l'exemple de l'application $\tau=z_1^2z_2^{-2}$ de C^2-0 dans l'espace projectif P^1 de coordonnées homogènes ζ_1,ζ_2, $\tau=\zeta_1\zeta_2^{-1}$, au voisinage d'un point $\zeta_1\neq 0$, $\zeta_2\neq 0$. Si τ est R-méromorphe et sans lacune, l'ensemble S des singularités de τ est analytique, de codimension 2 au moins; si τ est méromorphe, S est presque mince d'ordre 2 au moins. On étend aux applications méromorphes ou faiblement méromorphes des propriétés des applications R-méromorphes. Un espace complexe H est dit M-complet si chaque point a un

voisinage que des fonctions $f_1, \cdots, f_N$, méromorphes dans H, représentent holomorphiquement et biunivoquement dans un C^N. Pour les représentations τ dans un espace H qui est M-complet, méromorphie au sens de l'auteur entraine R-méromorphie si $\Sigma_\tau(P, L)$ contient toujours au moins un point. La méromorphie faible, la méromorphie, la R-méromorphie coincident si H est un espace projectif complexe ou un produit de tels espaces.

P. Lelong (Paris)

2062:

Stoll, Wilhelm. Über meromorphe Abbildungen komplexer Räume. II. Math. Ann. **136** (1958), 393–429.

Cette seconde partie (voir l'analyse précédente) repose sur un lemme qui étudie une situation géométrique particulière: soit E le sous-espace $z_{p+1} = \cdots = z_n = 0$ de C^n; G est un domaine de C^n coupé par E selon un domaine M; W est un domaine dans un plan complexe $C^1(w)$, $W_1 \subset W$ un domaine tel que $\overline{W}_1 \subset W$ et que $W - \overline{W}_1$ soit encore un domaine dans $C^1(w)$. On pose: $A = G - M$, $B = W \times A$, $S = W \times M$, $S_1 = \overline{W}_1 \times M \subset S$. N est un ensemble analytique dans B, de dimension p, régulier sur $S - S_1$, mais ayant un point singulier au moins dans S_1; on considère alors les plans $L(a)$ définis dans $W \times G$ par $z_j = a_j$, $1 \leqq j \leqq p$, et l'ensemble $F \subset M$ des $a \in M$ pour lesquels $N \cap L(a)$ n'a aucun point d'accumulation sur S. Dans ces conditions, à chaque $a \in M$ correspond $w \in S_1$, tel que N soit singulier en (w, a). De plus F est de mesure $2p$ dimensionnelle nulle.

Ce lemme conduit à deux énoncés permettant d'affirmer que l'application méromorphe τ est R-méromorphe, sous certaines hypothèses; l'image $\Sigma_\tau(S_R)$ de l'ensemble S_R où τ n'est pas R-méromorphe est supposée mince d'ordre $n-p$, S_R étant supposé mince d'ordre $p-1$, et presque mince d'ordre $p \geqq 2$. Si τ est faiblement méromorphe et $\Sigma_\tau(M)$ contenu dans un ensemble analytique de dimension 1, τ est R-méromorphe. La méromorphie entraine la R-méromorphie si, G et H étant des espaces à n dimensions complexes, S_R est dénombrable et si sur H, il existe $(n-1)$ fonctions méromorphes indépendantes. Cet énoncé permet des applications au cas où H est de dimension 2, non compact: si τ est sans lacune, τ est R-méromorphe. On applique les résultats à l'étude des modifications analytiques: pour $n=2$, toute modification $\mathcal{M}$ faiblement méromorphe est R-méromorphe; pour $n=3$ il en est de même si $\mathcal{M}$ et $\mathcal{M}^{-1}$ sont supposées méromorphes et si l'ensemble critique N est mince d'ordre 2.

P. Lelong (Paris)

2063:

Grauert, Hans; und Remmert, Reinhold. Komplexe Räume. Math. Ann. **136** (1958), 245–318.

Un revêtement analytique (ramifié) Y d'une variété analytique complexe X est la donnée $V = (Y, \eta, X)$ d'un espace connexe Y avec une projection η sur X, continue, propre, $\eta^{-1}(x)$ étant de plus discret; η est localement biunivoque hors d'un sous-ensemble analytique A, dit ensemble critique sur X; enfin $\eta^{-1}(A)$ ne sépare Y en aucun point localement. Un tel revêtement est dit algébroïde si ses ramifications sont celles d'une fonction algébroïde localement, c'est à dire si à tout $x \in X$ correspondent un voisinage U et une fonction sur $\eta^{-1}(U)$ qui soit de degré fini sur l'anneau des fonctions holomorphes dans U.

On sait que le terme "espace analytique" désigne selon les auteurs des notions différentes. Au sens de H. Behnke et K. Stein, il s'agit d'un espace dont tout point possède un voisinage homéomorphe à un revêtement $P = (Y, \eta, G)$ d'un domaine G de C^n. On appelle d'autre part espace analytique un espace qui a la structure d'un ensemble analytique normal (H. Cartan) ou d'un ensemble analytique quelconque (J. P. Serre). On désignera par espaces β ces derniers, c'est à dire les espaces R, annelés $[R, O(R)]$, dont chaque point a un voisinage topologiquement équivalent à un sous-ensemble analytique d'un domaine de C^n, et par β_n les espaces considérés par H. Cartan, où l'anneau O_x est anneau d'intégrité fermé dans son corps des quotients (espaces normaux). Le résultat essentiel du mémoire est l'identité des espaces α et β_n, tout espace α ayant localement la structure d'un revêtement algébroïde; on apporte de plus une étude des fonctions analytiques sur les espaces analytiques.

Si X est une variété analytique, $M \subset X$ un sous-ensemble analytique, si l'ensemble critique du revêtement $(S', \eta', X - M)$ se prolonge à travers M, S' se prolonge au-dessus de X; il en résulte que l'ensemble critique d'un revêtement est vide ou de codimension 1. Un ensemble analytique M^d défini dans un domaine $G^d \times G^{n-d}$ est dit étalé sur le domaine G^d à d dimensions complexes si la projection γ: $M \to G^d$ est propre, surjective, localement topologique en dehors d'un ensemble analytique $A \subset G^d$, $\gamma^{-1}(x)$ étant discret; la normalisation M^* d'un ensemble analytique M^d constitue un revêtement (M^*, γ^*, G^d); en tout point $x \in M^d$, il existe d'autre part un système d'axes dans lequel M^d est étalé, au voisinage de x, sur un domaine G^d d'un sous-espace; il en résulte qu'un espace β_n est du type α. Une fonction holomorphe sur un revêtement Y est par définition continue sur Y, holomorphe, au sens ordinaire, en tout y non projeté sur l'ensemble critique A: les notions de fonction holomorphe, d'ensemble analytique peuvent alors être étudiées sur les espaces α.

Sur un espace R du type β, le théorème de Riemann d'après lequel f, continue sur R, holomorphe sur $R - M$ (M étant un sous-ensemble analytique), est holomorphe sur R, n'est vrai que si R est du type β_n (normal). D'après K. Oka, on peut réaliser la normalisation M^* d'un ensemble analytique M au moyen d'un sous-ensemble analytique d'un domaine d'un C^N. On étudie alors, en utilisant des résultats de H. Cartan [Séminaire H. Cartan de l'École Norm. Sup. 1953/54; MR **19**, 577], l'espace vectoriel $H^0(R, O)$ des sections continues du faisceau O des germes de fonctions holomorphes sur un espace R de type β: $H^0(R, O)$ avec la topologie de la convergence compacte est complet, résultat non immédiat. Un théorème classique de Hartogs se généralise: $f(x_1, x_2)$ définie sur le produit $X_1 \times X_2$ de deux espaces β est holomorphe dès qu'elle l'est de x_1 et de x_2 séparément. Pour qu'une structure annelée O fasse de R, de type β, un espace β_n, il faut et il suffit que R porte une structure α de revêtements algébroïdes.

Pour établir l'identité des espaces α et β_n, il suffit alors de montrer que tout revêtement analytique est algébroïde. On se ramène par des équivalences analytiques à des types simplifiés de ramification. Localement un revêtement est équivalent à un revêtement (type a) du produit $G \times P^1(w)$ ramifié seulement au-dessus de $|w| < 1$. Dans un type b, l'ensemble critique A est formé des zéros d'un pseudo-polynome en z_1 à racines simples (au-dessus d'un point ordinaire de A, le revêtement a une ramification à groupe cyclique). Un revêtement a d'un domaine G^n est équivalent à un revêtement b de $(G^n - M) \times P^1$, M étant un

ensemble analytique de dimension $n-2$ au plus. De là on passe à un type c où les composantes irréductibles de A et l'ensemble A' des points non ordinaires de A n'ont que des points ordinaires: celui-ci se ramène par des éclatements à un type simplifié dont l'ensemble critique est $z_1z_2=0$ et pour lequel on établit le caractère algébroïde: tout revêtement a de $G^n \times P^1$ est alors algébroïde au dessus de $(G^n-M)\times P^1$. La théorie des faisceaux analytiques cohérents s'applique d'autre part sur les revêtements analytiques comme on le montre; elle est utilisée pour établir que le revêtement précédent est algébroïde partout. Un revêtement a est alors du type algébroïde; il en résulte que tout espace α a ses ramifications algébroïdes, et est donc un espace β_n, c'est à dire un espace analytique au sens de H. Cartan. *P. Lelong* (Paris)

SPECIAL FUNCTIONS

See also 1980, 2135, 2136, 2137, 2138, 2140, 2146, 2147, 2148a-b, 2162, 2167, 2311.

2064:

van der Blij, F.; and van Lint, J. H. On some special theta functions. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 508–513.

The authors define a theta function $\vartheta\left(\tau \middle| \begin{matrix} a \\ b \end{matrix}\right)$ in n variables based on a positive definite quadratic form and on two n-dimensional vectors a and b, and discuss its transformation under modular substitutions. By multiplying the theta function by η^{-n}, where $\eta(\tau)$ is the Dedekind modular form, they obtain forms $\varphi\left(\tau \middle| \begin{matrix} a \\ b \end{matrix}\right)$ of dimension 0. In a number of special cases, it is shown how to combine the φ-functions linearly to obtain a modular function.

J. Lehner (East Lansing, Mich.)

2065:

van Lint, J. H. On the multiplier system of the Riemann-Dedekind function η. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 522–527.

The Dedekind function

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$$

satisfies the transformation formula

$$\eta(L\tau) = \varepsilon(L)(c\tau+d)^{1/2}\eta(\tau)$$

for every modular substitution $L=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where v is a complicated 24th root of unity whose explicit dependence on a, b, c, d is a difficult problem first solved by Hermite and Dedekind in somewhat different forms. Their proofs were partly number-theoretic, partly function-theoretic. In recent years several purely function-theoretic proofs have appeared [cf. e.g., Rademacher, J. Indian Math. Soc. (N.S.) **19** (1955), 25–30; MR **17**, 15]. In the present paper, the author proves the Hermite formula—which he apparently attributes to Petersson!—by a combination of group theory and function theory.

The group-theoretic argument proceeds by observing that, since v^2 is a multiplier of dimension -1 on the modular group, it is a character of that group. It is next proved that every one-dimensional character on the modular group is of the form ζ^g, where ζ is a 12th root of unity, $g=f-3c$, c odd; $g=f+3d-3-3cd$, c even; and $f=(a+d)c-bd(c^2-1)$.

We now have $v(L)=\varepsilon(L)\exp \pi ig/12$, where $\varepsilon=\pm 1$. The determination of ε as a quadratic character is accomplished by a theorem of Hecke [Math. Ann. **119** (1944), 266–287; MR **6**, 173], the proof of which makes use of Eisenstein series and of the Gauss lemma in the theory of quadratic characters. *J. Lehner* (East Lansing, Mich.)

2066:

Ragab, F. M. A formula similar to Barnes' lemma. Ann. Polon. Math. **5** (1958), 149–152.

The following lemma is proved

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(s)\Gamma(\alpha-s)\Gamma(\beta-s)\Gamma(p-s)\Gamma(\alpha-p+s)\times\Gamma(\beta-p+s)(-1)^s ds$$
$$= 2^{-1-\frac{1}{2}p}e^{\frac{1}{4}i\pi p}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha-\tfrac{1}{2}p)\Gamma(\beta-\tfrac{1}{2}p)B(\tfrac{1}{2},\alpha+\beta-p),$$

where the path of integration separates the increasing sequence of poles from the decreasing sequence.

D. J. Hofsommer (Amsterdam)

2067:

Gautschi, Walter. Some elementary inequalities relating to the gamma and incomplete gamma function. J. Math. Phys. **38** (1959/60), 77–81.

The author gives lower and upper bounds of the form $c((x^p+c^{-1})^{1/p}-x)$ for $\exp(x^p)\int_x^\infty \exp(-t^p)dt$ in the range $p>1$, $0\leq x<\infty$; the respective values of c are 2 and $\{\Gamma(1+p^{-1})\}^{p/(p-1)}$. As it stands, the proof is only valid if p is an integer, but, in a correction, the author has indicated a modification which validates it for all $p>1$.

H. O. Pollak (Murray Hill, N.J.)

2068:

Nanjundiah, T. S. Van der Pol's expressions for the gamma function. Proc. Amer. Math. Soc. **9** (1958), 305–307.

Van der Pol has derived two new expressions for the gamma function $\Gamma(z+1)=\Pi(z)$, viz: the product

$$\Pi(z) = \left(\frac{z}{e}\right)^z\prod_{k=0}^{\infty}\frac{e_{z+k}}{e_k},$$

and its variant

$$\Pi(z) = \left(\frac{z}{e}\right)^z\lim_{n\to\infty}(2\pi n)^{1/2}\prod_{k=0}^{n}\frac{e_{z+k}}{e};$$

with the definitions $e_z=(1+1/z)^z$; $e_0=1$ [B. van der Pol, Canad. J. Math. **6** (1954), 18–22; MR **15**, 525]. The author points out two alternative methods of proof. In the first one both sides in the obvious relation

$$\left(1+\frac{1}{k}\right)\Big/\left(1+\frac{z}{k}\right) = \left(1+\frac{1}{z+k}\right)\Big/\left(1+\frac{z}{k+1}\right)$$

are raised to the power $z+k$. The result may be written

$$\left(1+\frac{1}{k}\right)^z\Big/\left(1+\frac{z}{k}\right) = \left(1+\frac{z}{k}\right)^{z+k-1}\Big/\left(1+\frac{z}{k+1}\right)^{z+k}\frac{e_{z+k}}{e_k}.$$

By using Euler's product $\Pi(z)=\prod_{k=1}^{\infty}(1+1/k)^z/(1+z/k)$ we find:

$$\Pi(z)=\left(\frac{z+1}{e}\right)^z\prod_{k=1}^{\infty}\frac{e_{z+k}}{e_k},$$

which is the same as the first result. The second result is easily derived from it by means of the elementary Stirling formula $n!\cong\left(\frac{n}{e}\right)^n(2\pi n)^{1/2}$ $(n\to\infty)$.

S. C. van Veen (Delft)

2069:

Nanjundiah, T. S. On a formula of A. C. Dixon. Proc. Amer. Math. Soc. **9** (1958), 308–311.

The author presents a very simple and elementary proof of the well-known formula of A. C. Dixon

$$\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}^3=(-1)^n\frac{(3n)!}{(n!)^3}\quad(n=0,1,2,\cdots)$$

[A. C. Dixon, Messenger of Mathematics **20** (1891), 79–80]. This proof is based upon the identity

$$\text{(A)}\quad\sum_{r=0}^{n}\binom{n}{r}^p\alpha^{n-r}\beta^r=\sum_{r=0}^{[n/2]}C_{n,r}{}^{(p)}\binom{n-r}{r}(\alpha+\beta)^{n-2r}(\alpha\beta)^r$$

and Vandermonde's identity

$${}_2F_1(-n,\beta;\gamma;1)=\frac{(\gamma-\beta)(\gamma-\beta+1)\cdots(\gamma-\beta+n-1)}{\gamma\cdot(\gamma+1)\cdots(\gamma+n-1)}.$$

By iteration it is found that

$$C_{n,r}{}^{(p)}=\sum_{0\le r_1\le\cdots\le r_p\le r}\binom{0}{r_1}\binom{r_2}{r_1}\cdots\binom{r_p}{r_{p-1}}\binom{r_p}{r}\binom{n}{r_1}\cdots\binom{n}{r_p}.$$

In particular

$$C_{n,r}{}^{(2)}=\binom{n}{r},\quad C_{n,r}{}^{(3)}=\binom{n+r}{r}\binom{n}{r}.$$

Taking $p=3$, $\alpha=1$, $\beta=-1$ and replacing n by zn in (A) we obtain Dixon's formula. As an analogous result the author mentions the formula

$$\sum_{r=0}^{n}(-1)^r\Big/\binom{m-1}{r}=\frac{m}{m+1}\left\{1+(-1)^n\Big/\binom{m}{n+1}\right\}\qquad(m>n\ge 0).$$

Special case:

$$m=2n+1,\ n\to 2n:\ \sum_{r=0}^{2n}(-1)^r\Big/\binom{2n}{r}=\frac{2n+1}{n+1}\qquad(n=0,1,2,\cdots).$$

S. C. van Veen (Delft)

2070:

Atoji, Masao. On the cubic spherical harmonics. J. Math. Phys. **38** (1959/60), 73–74.

The general expression for cubic spherical harmonics, invariant under the symmetry operation $m3m-O_h$ {misprinted O_n}, is obtained in the form

$$P_{2n}(\cos\theta)+\sum_m k_n{}^m P_{2n}{}^{4m}(\cos\theta)\cos 4m\phi,$$

where $1\le m\le\frac{1}{2}n$,

$$k_n{}^m=4(2n-4m)!P_{2n}{}^{4m}(0)/[2P_{2n}(0)+1](2n+4m)!,$$

and the P's are Legendre functions.

A. J. C. Wilson (Cardiff)

2071:

Popov, Blagoj S. Sur les fonctions de Legendre associées. C. R. Acad. Sci. Paris **248** (1959), 912–914.

In this note the author gives a formula which expresses the product of two associated Legendre functions in a series of associated Legendre functions. The formula is applied in the evaluation of some integrals.

D. J. Hofsommer (Amsterdam)

2072:

Popov, Blagoj S. L'évaluation explicite des expressions de Turán-Szegö. C. R. Acad. Sci. Paris **248** (1959), 2158–2159.

If

$$\Delta_n{}^r(x)=[D^rP_n(x)]^2-[D^rP_{n+1}(x)][D^rP_{n-1}(x)],$$

the author finds the inequality $\Delta_n{}^r(x)>0$, $|x|<1$, $n\ge r\ge 0$, which for $r=0$ reduces to an inequality derived by P. Turán.

D. J. Hofsommer (Amsterdam)

2073a:

Kuipers, L.; and Meulenbeld, B. Some properties of a class of generalized Legendre's associated functions. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 186–196.

2073b:

Kuipers, L.; and Meulenbeld, B. Related generalized Legendre's associated functions. Arch. Math. **9** (1958), 394–400.

2073c:

Kuipers, L. Generalized Legendre's associated functions (integral theorems, recurrence formulas). Monatsh. Math. **63** (1959), 24–31.

2073d:

Kuipers, L.; and Meulenbeld, B. Linear transformations of generalized Legendre's associated functions. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 328–329.

This series of papers concerns the solution of the differential equation

$$\text{(1)}\quad(1-z^2)w''-2zw'+\left[k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right]w=0,$$

which, for $m=n$, reduces to Legendre's associated equation. The reviewer investigated the solutions of (1) in his thesis, but confined himself mainly to real $|z|<1$. The authors' investigations concern complex values of z, k, m and n.

In their earlier paper [Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 436–450; MR **20** #5306], two solutions $P_k{}^{m,n}(z)$ and $Q_k{}^{m,n}(z)$ of (1) in the form of contour integrals were derived. For $m=n$ they reduce to Hobson's definitions of $P_k{}^m(z)$ and $Q_k{}^m(z)$. The contour for $P_k{}^{m,n}$ is a "clover" around the singular points $(1, z, \infty)$ and the contour for $Q_k{}^{m,n}$ a "clover" around $(-1, 1, \infty)$. These two functions we call the standard solutions of (1). From the integrals a representation was obtained in terms of hypergeometrical functions.

In #2073a, a contour around $(-1, z, \infty)$ is considered and the corresponding solution of (1) is expressed in the standard solutions. It is also shown that $P_{-1-k}^{m,n}$ is identical with $P_k^{m,n}$ and that $P_k^{m,-n}$ and $Q_k^{m,-n}$ are proportional to respectively $P_k^{m,n}$ and $Q_k^{m,n}$.

In #2073b, $Q_{-k-1}^{m,n}(z)$, $P_k^{-m,n}(z)$, $P_k^{n,m}(-z)$ and $Q_k^{n,m}(-z)$ are expressed in terms of the two standard solutions.

In #2073c, single-valued standard solutions are obtained by means of a cut $(-\infty, 1)$ along the x-axis in the z-plane, and real functions are defined in the segment $(-1, 1)$. These functions are shown to be orthogonal and their norm is evaluated. {In a related integral (eq. 18, case $m=n$) the factor $k+1$ in the denominator should be deleted.} Moreover this paper contains a few recurrency relations.

In #2073d a table is given which expresses the standard solutions in various ways in terms of hypergeometrical functions. The table is based on the well-known linear transformations of the hypergeometrical functions and on a relation derived in paper #2073c.

Summarizing, (1) is a Riemann equation with singularities in $(-1, +1, \infty)$ and hence is equivalent with the hypergeometrical differential equation. The theory in the above papers hence runs in part parallel with the theory of this equation and for another part is an extension of it. The advantage of the present treatment lies in the close parallel with the theory of Legendre functions.

{The reviewer regrets that the authors do not use the symbols $\mathfrak{P}$ and $\mathfrak{Q}$ for the functions with a complex argument.}

D. J. Hofsommer (Amsterdam)

2074:

Meijer, C. S. Expansions of generalized hypergeometric functions. Simon Stevin **31** (1957), 117–139. (Dutch)

This paper contains the subject-matter of a series of lectures delivered by the author at the Mathematical Centre at Amsterdam. The lectures are based on the functions ${}_p\phi_q(\alpha_s; \beta_t; z) = {}_pF_q(\alpha_s; \beta_t; z)/\prod_{r=1}^{q}\Gamma(\beta_r)$ rather then on the ${}_pF_q$. The expansions are divided into six classes; in each class the author starts with some known relations for ${}_0\phi_1$ (Bessel function), ${}_1\phi_1$ (Whittaker functions and Laguerre polynomials) and ${}_1\phi_2$ (common hypergeometric functions), and then gives a formula which contains these relations as special cases. It is shown that the six generalizations again are special cases of a super generalized formula, viz.

$$ {}_{p+k}\phi_{q+1}(\gamma_k, \alpha_p; \beta_q, \delta_1; \lambda z) = \sum_{r=0}^{\infty}\frac{1}{r!}\,{}_{k+1}\phi_1(-r, \gamma_k; \delta_1; \lambda) \times \prod_{s=1}^{p}(\alpha_s)_r(-z)^r\,{}_p\phi_q(\alpha_p+r; \beta_q+r; z). $$

Finally a proof of this formula is given.

D. J. Hofsommer (Amsterdam)

2075:

Saran, Shanti. Integral representations of Laplace type for certain hypergeometric functions of three variables. Riv. Mat. Univ. Parma **8** (1957), 133–143.

By using an integral representation of the Laplace type for $\Gamma(n)$ in the triple series that the author had previously used [Ganita **5** (1954), 77–91; MR **19**, 409] to define certain functions of three variables of the hypergeometric type, and interchanging the integral and summation signs, he obtains various single and double integral representations for his functions involving confluent hypergeometric functions. {Formula (2), p. 142 needs attention as it involves the letter p both as a parameter and as a variable of integration.}

R. G. Langebartel (Urbana, Ill.)

2076:

Shukla, Hari Shanker. On certain transformations of nearly-poised basic bilateral hypergeometric series of the type ${}_M\Psi_M$. Math. Z. **69** (1958), 195–201.

By using a transformation of Slater [Quart. J. Math. Oxford Ser. (2) **3** (1952), 73–80; MR **14**, 271], the author obtains the following relation between M nearly-poised series of the type ${}_M\psi_M$:

$$ \Pi\begin{bmatrix} q|c_1, q|c_2, \cdots, q|c_M, b_1, c_1b_1|c_2, c_1b_1|c_3, \cdots, c_1b_1|c_{M-1}, b_M; \\ q|a_2, q|a_3, \cdots, q|a_M, a_2, a_3, \cdots, a_M \end{bmatrix} \times {}_M\psi_M\begin{bmatrix} c_1, c_2, \cdots, c_{M-1}, c_M; \dfrac{a_2a_3\cdots a_M}{c_1c_2\cdots c_M} \\ b_1, c_1b_1|c_2, \cdots, c_1b_1|c_{M-1}, b_M \end{bmatrix} $$

$$ = \Pi\begin{bmatrix} a_2|c_1, a_2|c_2, \cdots, a_2|c_M, qb_1|a_2, qc_1b_1|a_2c_2, \cdots, \\ qc_1b_1|a_2c_{M-1}, qb_M|a_2 \\ a_2, q|a_2, a_2|a_3, \cdots, a_2|a_M, qa_2|a_2, \cdots, qa_M|a_2 \end{bmatrix} \times {}_M\psi_M\begin{bmatrix} qc_1|a_2, qc_2|a_2, \cdots, qc_{M-1}|a_2, qc_M|a_2; \dfrac{a_2a_3\cdots a_M}{c_1c_2\cdots c_M} \\ qb_1|a_2, qc_1b_1|a_2c_2\cdots qc_1b_1|a_2c_{M-1}, qb_M|a_2 \end{bmatrix} $$

$$ + \text{idem}\,(a_2; a_3, a_4, \cdots, a_M), $$

where

$$ \left|\frac{a_2a_3\cdots a_M}{c_1c_2\cdots c_M}\right| < 1 \quad \text{and} \quad \left|\frac{b_1{}^{M-1}c_1{}^{M-2}b_M}{a_2a_3\cdots a_Mc_2c_3\cdots c_{M-1}}\right| < 1. $$

Here the following notation is used:

$$ (a; n) = \prod_{r=0}^{n-1}(1-aq^r); \quad (a; 0) = 1. $$

$$ (a; -n) = (-1)^nq^{\frac{1}{2}n(n+1)}/a^n(q/a; n). $$

$|q| < 1$. $(a)_n = \prod_{r=0}^{n-1}(a+r)$; $(a)_0 = 1$; $(a)_{-n} = (-1)^n/(1-a)_n$.

$$ {}_r\psi_r\begin{bmatrix} a_1, a_2, \cdots, a_r; z \\ b_1, b_2, \cdots, b_r \end{bmatrix} = \sum_{n=-\infty}^{\infty}\frac{\prod_{s=1}^{r}(a_s; n)}{\prod_{s=1}^{r}(b_s; n)}z^n. $$

$$ \Pi\begin{bmatrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_r \end{bmatrix} = \prod_{n=0}^{\infty}\frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-b_1q^n)(1-b_2q^n)\cdots(1-b_rq^n)} $$

Idem $(a; b)$ means that the preceding expression is repeated with a and b interchanged. Several particular cases are considered. Further the author obtains certain relations involving well-poised and nearly-poised series.

As a special case the sum of a nearly-poised ${}_3\psi_3$ is obtained:

$$ {}_3\psi_3\begin{bmatrix} c_2{}^2|qa_3, qc_2|a_3, q|a_3; \dfrac{qa_3}{c_2{}^2} \\ q^2|a_3, c_2|a_3, qb_3|a_3 \end{bmatrix} = \Pi\begin{bmatrix} q^3|c_2{}^2, q|c_2, q, q, b_3, qa_3|c_2 \\ q^2|c_2, q^2a_3|c_2{}^2, qb_3|a_2, a_3|c_2, a_3, q^2|a_2 \end{bmatrix}. $$

S. C. van Veen (Delft)

2077:

Lesky, Peter. Unendliche orthogonale Matrizen und Laguerresche Matrizen. Monatsh. Math. **63** (1959), 59–83.

The Laguerre polynomials $L_\nu(x)$ can be defined from their orthogonality relation

$$\int_0^\infty e^{-x}L_\nu(x)L_\mu(x)dx = 0 \quad (\nu, \mu = 0, 1, \cdots; \nu \neq \mu);$$

if we add the condition that $L_\nu(x)$ is a polynomial of degree ν, this determines $L_\nu(x)$ apart from a factor not depending on x. The author replaces integration over $(0, \infty)$ by summation over 0, 1, 2, $\cdots$. In other words, he constructs polynomials $p_\nu(x)$, with degree ν, such that $\sum_0^\infty e^{-n}p_\nu(n)p_\mu(n) = \sigma_\nu\delta_{\nu\mu}$, where σ_ν is a normalization factor. He puts $p_\nu(n) = y_{\nu n}$, $r_{\nu n} = e^{-\frac{1}{2}n}y_{\nu n}\sigma_\nu^{-1/2}$. The construction succeeds by a variation method, giving $y_{\nu n} = e^n\Delta^\nu\left\{e^{-n}\binom{n}{\nu}\right\}$ (where Δ is defined by $\Delta f(n) = f(n+1) - f(n)$). In analogy to ordinary Laguerre polynomials, the author derives a generating function, a recursion formula with respect to ν, a difference equation with respect to n, and an expression of the type of Kogbetliantz' formula. Finally, it is shown that the infinite matrix $(r_{\nu n})$ is orthogonal (which includes that the rows form a complete system in the sequence space l_2). More generally, similar results are obtained for a generalization corresponding to the generalized Laguerre polynomials, based upon

$$\sum_{n=0}^\infty e^{-n}\binom{n+\alpha}{n}y_{\nu n}^{(\alpha)}y_{\tau n}^{(\alpha)} = \tau_\nu\delta_{\nu\mu}, \quad \alpha \geqq 0.$$

N. G. de Bruijn (Amsterdam)

2078:

Carlitz, Leonard. On some polynomials of Tricomi. Boll. Un. Mat. Ital. (3) **13** (1958), 58–64.

Durch Abänderung der nicht orthogonalen Tricomischen Polynome

$$l_n^{(\alpha)}(x) = \sum_{r=0}^n (-1)^r\binom{x-\alpha}{r}\frac{x^{n-r}}{(n-r)!}$$

[F. G. Tricomi, J. Analyse Math. **1** (1951), 209–231; MR 14, 466] in der Form $f_n^{(\alpha)}(x) = x^n l_n^{(\alpha)}(x^{-2})$ erhält der Verfasser für $\alpha > 0$ orthogonale Polynomsysteme, deren zugehörige Gewichtsfunktion in einfacher Weise durch eine Treppenfunktion gegeben ist.

K. Endl (Columbus, Ohio)

2079:

Fettis, Henry E. On the calculation of the function $j_0(z, \theta)$ for large values of "z". J. Math. Phys. **36** (1957), 279–283.

The author derives an asymptotic expansion for the function

$$j_0(z, \theta) = \int_0^\pi e^{iz\cos\varphi}\,d\varphi$$

which occurs frequently in physical problems involving wave phenomena. The final result is:

$$j_0(z, \theta) \sim \frac{\pi}{2}H_0^{(1)}(z) - e^{iz\cos\theta}\sum_{k=0}^\infty V_k(\cos\theta)\left(\frac{-i}{z}\right)^{1+k},$$

where $V_n(x) = d^n(1-x^2)^{-1/2}/dx^n = (1-x^2)^{-(n+1/2)}F_n(x)$, in which $F_n(x)$ is a polynomial of degree n which may be identified with the Gegenbauer function C_n^ν. A table of the first eight polynomials $F_n(x)$ is given by the author. It is shown that the given integral may be transformed into the non-oscillatory integral

$$j_0(z, \theta) = \frac{\pi}{2}H_0^{(1)}(z) + ie^{iz\cos\theta}\int_0^\infty \frac{e^{-zv}dv}{\sqrt{(v^2 - 2iv\cos\theta + \sin^2\theta)}},$$

from which the asymptotic expansion can be formally derived using Watson's lemma.

{Remark of the reviewer: It may be easily shown that

$$F_n(x) = n!x^nF\left\{-\frac{n}{2}, -\frac{n-1}{2}; 1; \frac{1}{x^2}\right\},$$

and so

$$j_0(z, \theta) \sim \frac{\pi}{2}H_0^{(1)}(z) + \frac{ie^{iz\cos\theta}}{z\sin\theta} \times \sum_{k=0}^\infty k!\left(\frac{-i\cos\theta}{z\sin^2\theta}\right)^k F\left\{-\frac{k}{2}, -\frac{k-1}{2}; 1; \frac{1}{\cos^2\theta}\right\}.$$

This expansion is only efficient if $|z\sin^2\theta|$ is large, in accordance with a remark by the author, that the accuracy attainable is greatly reduced if θ is close to zero. It may be remarked that in this case the given integral is strongly approximated by

$$(2/z)^{\frac{1}{2}}e^{iz}\int_0^{(2z)^{\frac{1}{2}}\sin\frac{1}{2}\theta} e^{-ix^2}dx$$

with an absolute error inferior to $\frac{1}{3}\sin^3\frac{1}{2}\theta$. The remaining integral can be expressed by Fresnel's integrals.}

S. C. van Veen (Delft)

ORDINARY DIFFERENTIAL EQUATIONS

See also 2108, 2109, 2171, 2193, 2383, 2550.

2080:

★Kaplan, Wilfred. Ordinary differential equations. Addison-Wesley Series in Systems Engineering. (Available in U.S.A. and Canada from) Addison-Wesley Publishing Co., Inc., Reading, Mass. xv + 534 pp. $8.50.

This book would serve as a very good text for a one year introductory course to the theory of ordinary differential equations. The subject is presented from the point of view of functional analysis, with a strong emphasis on the geometrical aspects of the subject. The modern input-output terminology of the engineer is extensively used. Problems and answers, as well as an up-to-date list of references for further study, are given at the end of each chapter. The chapter content is as follows.

Chap. I: Basic ideas and definitions are presented with care. Existence theorems are stated. A simple step-by-step integration method is given. Chap. II: Equations of the first order are treated. The standard types are presented with more than usual care with regard to rigor. Applications are discussed. Chap. III: Properties of solutions of linear equations of the first order are studied in terms of the input-output vocabulary of the engineer. Responses to standard and general inputs are described and Fourier series are brought into play. Chap. IV: An introduction to the general theory of linear equations of any order is presented and equations with constant coefficients are treated in detail. Operational methods are briefly presented. Chap. V: Properties of solutions of linear equations

are discussed in more detail with attention to responses, steady and transient solutions, stability, etc. More applications are introduced. Chap. VI: Systems of simultaneous linear equations are studied with applications to mechanical and electrical networks. An appendix introduces the use of matrices in solving systems with constant coefficients. Chap. VII: Exact differential equations of first and higher orders are treated. Chap. VIII: Equations of higher than the first degree are examined and singular solutions are obtained. Chap. IX: Series solutions are treated in considerable detail. Chap. X: Numerical methods of solution are briefly presented. Chap. XI: Phase plane analysis is elaborated. Chap. XII: Existence and uniqueness theorems are given and proved with rigor.

H. L. Turrittin (Minneapolis, Minn.)

2081:

Corduneanu, C. Sur les systèmes différentiels de la forme $y'=A(x,y)y+b(x,y)$. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 45–52. (Romanian and Russian summaries)

Si considera il sistema differenziale $y'=A(x,y)y+b(x,y)$; y e $b(x,y)$ sono vettori ad n componenti, $A(x,y)$ è una matrice di rango n. In opportune ipotesi di continuità per A e b e supponendo che i moduli degli autovalori della matrice $\frac{1}{2}(A+A^*)$ ed il modulo di b si lascino maggiorare da funzioni continue della sola x, l'A. dimostra l'esistenza di almeno un integrale del sistema che assume valori prefissati in un punto ed è definito in un dato intervallo. Vengono indicate delle limitazioni per gli integrali trovati e data una condizione sufficiente per l'unicità.

G. Fichera (Madison, Wis.)

2082:

Cameron, R. H. Erratum. Proc. Amer. Math. Soc. **9** (1958), 1000.

To same Proc. **8** (1957), 834–840 [MR **19**, 747].

2083:

Luxemburg, W. A. J. On the convergence of successive approximations in the theory of ordinary differential equations. Canad. Math. Bull. **1** (1958), 9–20.

Let $x'=f(t,x)$ be such that f is continuous and satisfies the Krasnosel'skiĭ-Kreĭn condition

$$|f(t,x_1)-f(t,x_2)| \leq k(t-t_0)^{-1}|x_1-x_2|,$$

$$|f(t,x_1)-f(t,x_2)| \leq A|x_1-x_2|^{\alpha},$$

$0<\alpha<1$, $k(1-\alpha)<1$, then the successive approximations

$$x_n(t) = x_0+\int_{t_0}^{t} f(\lambda, x_{n-1}(\lambda))d\lambda$$

converge uniformly. Two proofs of the theorem are given.

R. T. Herbst (Winston-Salem, N.C.)

2084:

Brauer, Fred. A note on uniqueness and convergence of successive approximations. Canad. Math. Bull. **2** (1959), 5–8.

The results obtained by W. A. J. Luxemburg [see above review] are generalized for systems of differential equations.

R. T. Herbst (Winston-Salem, N.C.)

2085:

Reznikovskiĭ, P. T. The region of convergence of a power series representing the solution of a differential equation. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 145–150. (Russian)

The author considers an analytic differential equation of the form (1) $dx/dt=X(x,t,\mu)$, where μ is a small parameter and $X(x,t,0)\equiv 0$. The solution of (1) with $x(t_0)=a$, is given by a series (2) $x=a+x_1\mu+x_2\mu^2+\cdots$, convergent in some interval $|t-t_0|<T$, $T>0$, for μ small. If $\Phi(x,\mu)$ is a majorant of $X(x,t,\mu)$ in a neighborhood of t_0 and $\Phi(x,0)\equiv 0$, then $z=a+T\Phi(z,\mu)$, $a=|\alpha|$, is a majorant of (2), and actually the series $z=a+z_1\mu+z_2\mu^2+\cdots$ is convergent for $T\Phi(z,\mu)/(z-a)\leq q<1$ and any $q<1$ [N. V. Bugaev, Mat. Sb. **22** (1901), 219–224]. Thus $T\leq q\Psi(z,\mu)$, with $\Psi(z,\mu)=(z-a)/\Phi(z,\mu)$. An evaluation of T is thus $T=\max \Psi(z,\mu)$, or $T=\Psi(z_0,\mu)$ for $\Psi_z(z_0,\mu)=0$. Examples are given which show that this evaluation of T can be better than the analogous one obtained by a Cauchy majorant.

L. Cesari (Baltimore, Md.)

2086:

Morimoto, Hiroshi. On the perturbation of the linear system with periodic coefficients. Bull. Univ. Osaka Prefecture Ser. A **6** (1958), 11–14.

This paper is concerned with the system of equations

$$\frac{dz_i}{dt} = f_i(t)+\sum_j f_{ij}(t)z_j+\lambda g_i(z_1,\cdots,z_n,t,\lambda) \quad (i=1,\cdots,n),$$

where the functions f_i, f_{ij}, g_i are periodic with respect to t, with a common period T, and satisfy certain simple continuity conditions. It is assumed that when λ is zero there exists a degenerate periodic solution with period T; and conditions are sought which are sufficient for the existence of a periodic solution when $|\lambda|$ is small but not zero. Poincaré has discussed the problem under the assumption that the right-hand members of the equations are analytic, but without assuming that the system reduces, for $\lambda=0$, to a linear system. [H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Gauthier-Villars, Paris, 1892; Ch. III.] The methods used in the present discussion are fundamentally similar to those of Poincaré; and, although some of the statements are not entirely clear, it appears that the results obtained are essentially equivalent to his.

L. A. MacColl (New York, N.Y.)

2087:

Štelik, V. G. On the solutions of a linear system of differential equations with almost periodic coefficients. Ukrain. Mat. Ž. **10** (1958), 318–327. (Russian. English summary)

The author considers m-dimensional systems $x'=[Q_0+Q(t)]x$ where Q_0 is constant and $Q(t)$ satisfies the condition (Q$^+$): $Q(t)=\sum_1^\infty Q_k \exp i\nu_k t$, $\nu_k\geq\nu_1>0$, and $Q_k=(q_{ij}^k)$ with $\sum_k |q_{ij}^k|<\infty$. He proves that the fundamental matrix $X(t,t_0)$ with $X(t_0,t_0)=I$ has the form $X(t,t_0)=\exp[Q_0(t-t_0)]G(t,t_0)$ where G is almost periodic and satisfies (Q$^+$), provided either $Q_0Q_k=Q_kQ_0$ for $k\geq 1$ or the following condition holds: $Q_0=\sum_0^{2s}(\mu+i\lambda_j)I_{m_j}$, where $\lambda_0=0$, $\lambda_j=\gamma_j$ for $1\leq j\leq s$, $\lambda_j=-\gamma_{j-s}$ for $s+1\leq j\leq 2s$, $\sum m_j=m$, and I_{m_j} is the $m\times m$ matrix with ones in the main diagonal in the rows $\sum_0^{j-1} m_i+p$ for $1\leq p\leq m_j$ and zeros

elsewhere, and either $\nu_1 > 2\max|\lambda_j|$ or $\nu_1 = 2\max|\lambda_j|$ and $I_{m_s}Q_1I_{m_s}=0$. Under the same hypotheses and under the assumption that $Q(t)=\sum_{k\neq 0}Q_k\exp i\nu_k t$ with $-\nu_k=\nu_{-k}$ and $Q_k=(q_{i,j}{}^k)$ with $\sum_{k\neq 0}|q_{i,j}{}^k|<\infty$, he also derives criteria for the stability and instability; these include systems $x'=[Q_0+\varepsilon Q(t)]x$ in the critical case.

H. A. Antosiewicz (Los Angeles, Calif.)

2088:

Ráb, Miloš. Asymptotische Eigenschaften der Lösungen der Differentialgleichung $y''+A(x)y=0$. Czechoslovak Math. J. 8 (83) (1958), 513–519. (Russian summary)

In der vorliegenden Arbeit werden asymptotische Formeln der Lösungen der Differentialgleichung $y''+A(x)y=0$ abgeleitet. Aus ihnen folgt eine einfache notwendige und hinreichende Bedingung dafür, daß alle Lösungen der angeführten Differentialgleichung für $x\to\infty$ gegen Null streben. *Zusammenfassung des Autors*

2089:

Čerkasov, A. N. A note on linear differential equations of second order. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 3, 13–17. (Russian)

In the first part of this paper the author studies the Riccati equation (1) $w'-w^2+p(x)w-q(x)=0$ for bounded $p(x)$, $q(x)$, $x\geqq x_0$, and $q(x)\leqq N<0$. By comparison analysis it is shown that (1) has a continuous, bounded, strictly negative solution, provided

$$m/2-(m^2/4-n)^{1/2} < M/2-(M^2/4-N)^{1/2},$$

where $M=\sup p(x)$, $m=\inf p(x)$, $N=\sup q(x)$, $n=\inf q(x)$, $x\geqq x_0$. In the second part of the paper the author considers the linear equation (2) $y''+p(x)y'+q(x)=0$ under the same assumptions for $p(x)$, $q(x)$. By reduction of (2) to (1) and convenient analysis it is shown that through every point $(\bar{x}_0,\bar{y}_0)$, $\bar{x}_0\geqq x_0$, there passes a solution $\varphi(x)$ of (2) with $\varphi(x)\to 0$ as $x\to+\infty$. *L. Cesari* (Baltimore, Md.)

2090:

Kostmarov, D. P. Exponential order of growth of the solutions of systems of linear differential equations. Vestnik Moskov Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 1, 33–38. (Russian)

The author considers linear differential systems

$$(*)\qquad \frac{dw_i}{dz}=\sum_j a_{ij}(z)w_j \quad (i=1,\cdots,n)$$

with coefficients $a_{ij}(z)$ holomorphic in a region

$$R\leqq|z|<+\infty$$

of the complex z-plane. The author evaluates the order of growth of the solutions as compared with the order of growth of the coefficients as $z\to\infty$. If $[w_{ij}(z)]$ is any fundamental system of solutions of (*), then the rank r, or the exponential order of growth of the solutions of system (*), is defined by

$$r=\sup r(\varphi)\quad(-\infty<\varphi<+\infty),\qquad r(\varphi)=\max[\bar{r}(\varphi),0],$$

$$\bar{r}(\varphi)=\limsup(\ln\rho)^{-1}[\ln|\ln\sum_{ij}|w_{ij}(\rho e^{i\varphi})|\,|]$$

as $\rho\to+\infty$. The rank r does not depend on the chosen fundamental system. The order of growth of the coefficients of (*) is defined as the number

$$m=\max_{ij}m_{ij},\qquad m_{ij}=\sup m_{ij}(\varphi)\quad(-\infty<\varphi<+\infty),$$

$$m_{ij}(\varphi)=\max[\bar{m}_{ij}(\varphi),0],$$

$$\bar{m}_{ij}(\varphi)=\limsup(\ln\rho)^{-1}\ln|a_{ij}(\rho e^{i\varphi})|.$$

The main result is: (I) For any system (*) we have $0\leqq r\leqq\max(0,1+m)$. This statement contains a Lyapunov theorem for the case of bounded coefficients. The author proves also another result slightly stronger than (I).

L. Cesari (Baltimore, Md.)

2091:

Fage, M. K. Operator-analytic functions of one independent variable. Trudy Moskov. Mat. Obšč. **7** (1958), 227–268. (Russian)

Let $L=D^n+p_{n-1}(x)D^{n-1}+\cdots+p_0(x)$ be a differential operator ($D=d/dx$) whose coefficients are complex-valued continuous functions of a real variable $x\in(a,b)$. A function $f(x)$ is L-analytic in (a,b) if $D^rL^qf(x)$ exist and are continuous for all integral q and $r=0,1,\cdots,n-1$, and are bounded in each subinterval $[\alpha,\beta]\subset(a,b)$ by $C^{qn+r}\cdot(qn+r)!$, C being a constant which depends only on α, β; any eigenfunction of L is obviously L-analytic; if $L=D$, L-analyticity reduces to ordinary analyticity; this is also the case if the $p_i(x)$ are (ordinary) analytic functions. Let $x_0\in(a,b)$, and $f_m(x,x_0)$, for $m=0,1,\cdots,n-1$, be the solutions of $Ly=0$ with $D^sf_m(x,x_0)|_{x=x_0}=\delta_{sm}$, $s=0,1,\cdots,n-1$, and for $m\geqq n$, be the solutions of $Ly=f_{m-n}(x,x_0)$ with $D^sf_m(x,x_0)|_{x=x_0}=0$, $s=0,1,\cdots,n-1$; the f_m are L-analytic and form what is called an L-basis at x_0; if $L=D$ they reduce to the usual $(x-x_0)^m/m!$. A sequence $\{a_m\}$ is A-majorized if $|a_m|\leqq C^m\cdot m!$ for a suitable C; for any given such sequence and any L-basis, $\sum a_mf_m(x,x_0)$ is called an L-series (they reduce to ordinary convergent power series if $L=D$); any L-series converges in an appropriate neighborhood of x_0, its sum is an L-analytic function $f(x)$, it may be termwise D^rL^q-differentiated and $a_m=D^rL^qf(x)|_{x=x_0}$ for $m=qn+r$, $0\leqq r\leqq n-1$ (the series is the L-Taylor series of f). The theory of analytic functions is thus satisfactorily extended to L-analytic functions. The sequence of A-majorized sequences $\{a_m{}^{(k)}\}$ [of L-analytic functions $\{f_k(x)\}$] converges regularly if they are equi-A-majorized [if $|D^rL^qf_k(x)|\leqq C^{qn+r}\cdot(qn+r)!$ in each sub-interval] and if they converge termwise $a_m{}^{(k)}\to a_m$ [if $D^rL^qf_k(x)$ converge]; this induces a topology in the algebra A_I of A-majorized sequences [A_{L,x_0} of L-analytic functions in the neighborhood of x_0] (the algebra operations are defined in a natural way in correspondence with the elementary operations with L-series). The main theorems are: any two differential operators L, M of the same order are equivalent in the following sense: there is an isomorphism $T: A_{M,x_1}\to A_{L,x_0}$ such that $LT=TM$; if Λ is the shift operator in A_I, $\Lambda(\{a_0,a_1,\cdots\})=\{a_1,a_2,\cdots\}$, any nth order differential operator L is equivalent to the nth power of Λ (as a corollary, $L^{1/n}$ exists). The theory can be extended to (M,L)-series of functions of two variables $F(w,x)$, the operator M acting on w and L acting on x; Cauchy's problem for the partial differential equation $MF(w,x)=LF(w,x)$ may then be solved even in the case of operators L, M of different orders having non-analytic coefficients. The theory thus developed is closely related to, and generalizes and unifies, previous results of J. Delsarte [J.

Math. Pures Appl. (9) **17** (1938), 213–231; Acta Math. **69** (1938), 259–317], A. Ya. Povzner [Mat. Sb. (N.S.) **23** (**65**) (1948), 3–52; MR **10**, 299], V. A. Marčenko [Trudy Moskov. Mat. Obšč. **1** (1952), 327–420; **2** (1953), 3–83; MR **15**, 315] and others. *J. L. Massera* (Montevideo)

2092:

Marčenko, V. A.; and Rofe-Beketov, F. S. Expansion in characteristic functions of non-selfadjoint singular differential operators. Dokl. Akad. Nauk SSSR **120** (1958), 963–966. (Russian)

Consider the problem (*) $-y''+qy=\lambda^2 y, y'(0)-Ay(0)=0$ on $0\leq x<\infty$. Here q is a complex-valued function, summable on every finite subinterval of $0\leq x<\infty$, and A is any complex number. In general this problem is non-self-adjoint. The purpose of the paper is to state some results which are extensions of known ones for the self-adjoint case. By Z is denoted the set of all even entire functions of finite order which are summable on the real axis. A sequence $F_n\in Z$ is said to converge to zero if $\lim \int_0^\infty |F_n(\lambda)|d\lambda=0$ $(n\to\infty)$, and the orders σ_n are bounded. The continuous linear functionals on Z are denoted by $T(Z)$. If $T\in T(Z)$ the value of T at $F\in Z$ is denoted by (F, T). The set of all $F\in Z$ of order $\leq\sigma$ which are in $L^2(-\infty,\infty)$ is denoted by W_σ^2. Let $\omega(=\omega(\lambda,x))$ be the solution of the differential equation in (*) satisfying $\omega(\lambda,0)=1$, $\omega'(\lambda,0)=A$.

Theorem 1: Each problem (*) determines an $R\in T(Z)$ such that $(E_fE_g, R)=\int_0^\infty f(x)g(x)dx$, where f, g are any functions in $L^2(0,\infty)$ vanishing outside compact subsets of $0\leq x<\infty$, and

$$E_f(\lambda)=\int_0^\infty f(x)\omega(\lambda,x)dx,\quad E_g(\lambda)=\int_0^\infty g(x)\omega(\lambda,x)dx.$$

If $E_f\in L^1(0,\infty)$, then $f(x)=(E_f\omega_x, R)$, where $\omega_x(\lambda)=\omega(\lambda,x)$. The functional R is called the spectral function of (*). Theorem 2. In order that an $R\in T(Z)$ be a spectral function of some problem (*) with a q having n derivatives $(n\geq 0)$, it is necessary and sufficient that (a) the function ϕ defined by $\phi(x)=((1-\cos\lambda x)/\lambda^2, R)$ have $n+3$ derivatives for $x\geq 0$, and $\phi'(+0)=1$, (b) if $F\in W_\sigma^2$ and $(FX, R)=0$ for all $X\in W_\sigma^2$, then F is identically zero. The function q and A are uniquely determined by R.

It is stated that the methods can be used to treat the case when $q(x)$ and A are bounded operators in some Banach space B, with q continuous in x, $x\geq 0$. In this case the solution $\tilde\omega$ of $-v''+vq=\lambda^2 v$, $v(0)=I$, $v'(0)=A$ is needed, as well as the transform $\tilde E_f(\lambda)=\int_0^\infty \tilde\omega(\lambda,x)f(x)dx$. If B is a separable Hilbert space the operator functions can be written in matrix form, $q(x)=(q_{ik}(x))$, etc. Theorem 3: Every operator problem (*) determines a spectral matrix $R=(R_{ik})$, $R_{ik}\in T(Z)$, such that $(E_fR\tilde E_g)=\int_0^\infty f(x)g(x)dx$. Here f, g are continuous operator-valued functions vanishing outside compact subsets,

$$(E_fR\tilde E_g)=((E_f{}^{im}\tilde E_g{}^{nk}, R_{mn}))$$

(summation on m, n).

A result corresponding to Theorem 2 is also stated. For the self-adjoint operator case, the following result is asserted. If $q^*(x)=q(x)$, $A^*=A$, then the spectral matrix R is an operator measure, i.e.,

$$\int_0^\infty f(x)g^*(x)dx=\int_{-\infty}^\infty E_f(\sqrt\lambda)\,[d\rho(\lambda)]\,E_g{}^*(\sqrt\lambda),$$

where ρ is a non-decreasing operator-valued function.

E. A. Coddington (Los Angeles, Calif.)

2093:

Fage, M. K. Two spectral distribution matrices. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1 (79), 207–210. (Russian)

Explicit computation of the spectral matrices for the operator D^n on $L_2(-\infty,+\infty)$ with respect to two different sets of initial conditions at $x=0$ (D denotes the operator id/dx). For the special case of $D^2=-d^2/dx^2$ this provides a correction of an erroneous expression given earlier by Bloh [Dokl. Akad. Nauk SSSR **92** (1953), 209–212; MR **15**, 708]. The method consists in comparing two different integral expressions for the kernel of the resolution of identity of D^n: (1) the expression in terms of the integrals of certain eigenfunctions that defines the spectral matrix [see, for instance, Stone, *Hilbert space*, Amer. Math. Soc. Colloq. Publ., New York, 1932; Th. 10.22; at least for the case $n=2$]; and (2) the explicit integral representation of the kernel obtained from the fact that D is the Fourier-Plancherel transform of multiplication by x. *A. Brown* (Houston, Tex.)

2094:

Coddington, Earl A. Generalized resolutions of the identity for symmetric ordinary differential operators. Ann. of Math. (2) **68** (1958), 378–392.

Using Krein's theory [Ahiezer and Glazman, *Teoriya lineĭnyh operatorov v gil'bertovom prostranstve*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad; translated as *Theorie der linearen Operatoren im Hilbert-Raum*, Akademie-Verlag, Berlin, 1954; MR **13**, 358; **16**, 596] of generalized resolvents of symmetric operators and their generalized spectral measures, the author develops the Weyl-Kodaira theory for symmetric ordinary differential operators L in this more general setting. He shows that the generalized resolvent $R(l)$ is an integral operator of Carleman type having the kernel $K(x,y,l)$ with the property that $\partial^{j+k-2}K/\partial x^{j-1}\partial y^{k-1}$, $j, k=1\cdots n$, are continuous in x, y, l and analytic in l on any region for which $\Im(l)\neq 0$, except for $x=y$, when j or k is n. He derives a spectral matrix measure $\|\rho_{j,k}\|$ such that, if $f\in L(a,b)$ and vanishes outside a closed bounded subinterval of (a,b) and μ, λ are points of continuity of the generalized spectral measure F, then

$$F([\lambda,\mu))=\int_\Delta \sum_{j,k=1}^n s_k(x,\nu)\hat f_j(\nu)d\rho_{jk}(\nu),$$

where $s_k(x,\nu)$ are solutions of $(L-\nu)s_k=0$ chosen in the usual way, and $\hat f_j=(f(\gamma), s_j(\gamma))$. The expression for $\hat f_j$ converges in $L^2(\rho)$, and $\|f\|_{L^2}=\|f\|_{L^2(\rho)}$, using the usual definitions. *František Wolf* (Berkeley, Calif.)

2095:

Riekstyn'š, Ê. Ya. On the method of tabulated functions. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 65–86. (Russian. Latvian summary)

A résumé of the literature and history of the method of R. E. Langer for the asymptotic solution of differential equations with turning points is given. The main body of the paper is devoted to a unification and some extensions and modifications of known results. In particular, the equation $v''+[\lambda^2 t^\alpha+at^{-2}+t^{\beta-2}g(t)]v=0$ is considered where g is bounded, $\alpha>-2$, $0<\beta\leq 2$ and $\beta-\alpha=1$, 2, or 3; and

the results of Langer [Trans. Amer. Math. Soc. **37** (1935), 397–416] are refined in these cases.

N. D. Kazarinoff (Ann Arbor, Mich.)

2096a:

Bihari, Imre. Extension of certain theorems of the Sturmian type to nonlinear second order differential equations. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 13–20. (Hungarian and Russian summaries)

2096b:

Bihari, Imre. Ausdehnung der Sturmschen Oscillations- und Vergleichssätze auf die Lösungen gewisser nichtlinearer Differentialgleichungen zweiter Ordnung. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 3/4, 159–173. (Hungarian and Russian summaries)

In these two papers the author considers nonlinear systems closely related to linear. Related to the linear differential equation (1) $(P(x)y')'+Q(x,\lambda)y=0$ and the linear systems (2) $y'=P_\nu(x)y+Q_\nu(x)z$, $z'=R_\nu(x)y+S_\nu(x)z$ $(\nu=1,2)$, consider the nonlinear equation (3) $(P(x)y')'+Q(x,\lambda)f(y,y')=0$, and the nonlinear systems (4) $y'=P_\nu(x)g(y,z)+Q_\nu(x)G(y,z)$, $z'=R_\nu(x)h(y,z)+S_\nu(x)H(y,z)$ $(\nu=1,2)$. In (3) and (4), f, g, G, h, and H are defined for all values of their variables, satisfy Lipschitz conditions in any bounded region, and are homogeneous of degree 1 in their variables. Also $\operatorname{sgn} f=\operatorname{sgn} g=\operatorname{sgn} h=\operatorname{sgn} y$, and $\operatorname{sgn} G=\operatorname{sgn} H=\operatorname{sgn} z$. As a result of the homogeneity, the nonlinear systems have the property that if y [(y,z)] is a solution of (3) [(4)], then so is cy [(cy,cz)] for any constant c.

In the first paper the author extends various Sturmian oscillation and comparison theorems from (1) and (2) to (3) and (4) respectively. For example, in (3) let $P(x)$ be positive and continuous on $a\leq x\leq b$, let $Q(x,\lambda)$ be continuous for $a\leq x\leq b$ and all λ, strictly increasing in λ, and such that $Q(x,\lambda)\to\pm\infty$ as $\lambda\to\pm\infty$ uniformly in x on $a\leq x\leq b$, and let boundary conditions $\alpha y(a)+\beta y'(a)=0$, $\gamma y(b)+\delta y'(b)=0$ be imposed. Then there exists an infinite increasing sequence of eigenvalues for (3), $\lambda_n\to+\infty$, and the eigenfunction $\varphi_n(x)$ has n zeros in $a<x<b$.

The author then considers the case of (3) with $P(x)\equiv 1$, and $f(y,y')=y^3/(y^2+y'^2)$. For $\lambda_1<\lambda_2$, let $Q(x,\lambda)$ satisfy, in addition to the conditions mentioned above, $Q(x,\lambda_2)>Q(x,\lambda_1)>0$, and $Q(x,\lambda_2)-Q(x,\lambda_1)\geq Kx$ for some positive K as $x\to 0+$. Then if $y(x,\lambda)$ is the solution of (3), $y(0,\lambda_1)=y(0,\lambda_2)=0$, $y'(0,\lambda_1)\geq y'(0,\lambda_2)>0$, it is shown that $y(x,\lambda_1)>y(x,\lambda_2)$ for all $x>0$ up to the first positive zero of $y(x,\lambda_2)$.

For this special P and f it is also shown that if $Q(x,\lambda)$ is strictly increasing in x, the distances between successive zeros and extrema of solutions of (3) decrease as x increases.

In the second paper equation (3) is considered with Q independent of λ. P and Q are continuous and P is positive on $a\leq x\leq b$. If v is any solution of (3), and φ_1 and φ_2 are differentiable on $a\leq x\leq b$ and satisfy (5) $\varphi_1'\varphi_2-\varphi_2'\varphi_1+[\varphi_1^2+\varphi_2^2Qf(\varphi_2P,\varphi_1)]/P\neq 0$ on $[a,b]$, then $\Phi(x)=\varphi_1v-\varphi_2Pv'$ cannot have an infinity of zeros on $[a,b]$. It is shown further that for $f(y,y')=y^3/(y^2+y'^2)$, and under rather complicated hypotheses, the zeros of $\Phi_1=\varphi_1v_1-\varphi_2Pv_1'$ and $\Phi_2=\varphi_1v_2-\varphi_2Pv_2'$ interlace, where v_1 and v_2 are any linearly independent solutions of (3). A further result on interlacing of zeros for this special $f(y,y')$ is obtained for $\Phi=\varphi_1v-\varphi_2Pv'$ and $\Psi=\psi_1v-\psi_2Pv'$ where v is any solution of (3), φ_1, φ_2, ψ_1, and ψ_2 are differentiable, $\varphi_1\psi_2-\psi_1\varphi_2\neq 0$, and the φ's and ψ's both satisfy (5) above.

W. S. Loud (Minneapolis, Minn.)

2097:

Greguš, M. Bemerkungen zu den oscillatorischen Eigenschaften der Lösungen der Differentialgleichung dritter Ordnung. Acta Fac. Nat. Univ. Comenian. **3**, 23–28 (1958). (Slovak. Russian and German summaries)

The author concerns himself with the equation

$$y'''+2A(x)y'+[A'(x)+b(x)]y=0, \tag{1}$$

assuming $A'(x)$, $b(x)$ continuous on $(-\infty,\infty)$ and $A(x)>0$, $b(x)\geq 0$ on $(-\infty,\infty)$. In addition it is assumed that on no interval does $b(x)$ vanish identically.

His results may be summarized as follows. A. If $y_1(x)$, $y_2(x)$ are two independent solutions of (1) which are oscillatory and have interlacing zeroes on $(-\infty,\infty)$, then their Wronskian $\omega(x)$, which is a solution of the equation adjoint to (1), namely

$$y'''+2A(x)y'+[A'(x)-b(x)]y=0, \tag{2}$$

does not vanish on $(-\infty,\infty)$. B. Moreover, all solutions of (1) of the form $c_1y_1+c_2y_2$ are also oscillatory on $(-\infty,\infty)$, satisfy the equation

$$(y'/\omega)'+(2A/\omega+\omega''/\omega^2)y=0, \tag{3}$$

and any independent pair of them have interlacing zeroes. C. If $y(x)$ is a solution of (1) which vanishes at the endpoints of the interval $x_1\leq x\leq x_2$ but not in its interior, then every solution $\bar{y}(x)$ of (1) which vanishes at the endpoints of $(\bar{x},x_2)$, where $x_1\leq\bar{x}\leq x_2$, must be non-vanishing in the interior of $\bar{x}\leq x\leq x_2$ and also in the interior of $x_1\leq x\leq\bar{x}$. D. A solution $y(x)$ of (1) with zeroes at x_1, x_2 $(x_1<x_2)$ will have no zero between x_1 and x_2 if and only if $0<x_2-x_1<x_2-x_s$, where x_s is the first zero of $\omega(x)$ to the left of x_2 and $\omega(x)$ is a solution of (2) having a double zero at x_2.

The proofs depend on Mammana's identity for (1) and a previous paper by the author [Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 73–85; MR **17**, 737].

J. F. Heyda (Cincinnati, Ohio)

2098:

Myškis, A. D.; and Hohryakov, A. Ya. Surging dynamical systems. I. Singular points in the plane. Mat. Sb. N.S. **45 (87)** (1958), 401–414. (Russian)

After some very general definitions, the following situation is presented: n systems of differential equations f_α:

$$\begin{aligned}\dot{x}&=X_\alpha(x,y)\\ \dot{y}&=Y_\alpha(x,y)\end{aligned}\qquad(\alpha=1,\cdots,n)$$

of class C^1 in a neighborhood of the origin; a disjoint family Γ_β $(\beta=1,\cdots,m)$ of half-open arcs, each of which has the origin for an endpoint; and a set of m functions $\varphi_\beta(\alpha)$ on the integers from 1 to n to themselves. Sets of trajectories are defined by following a trajectory of f_α until it hits a set Γ_β, where $f_{\varphi_\beta(\alpha)}$ takes over. The qualitative behavior—stability of focus, etc.—is discussed by piecing together information from the separate vectors. T. Vogel treated the case $n=2$, $\varphi_\beta(1)=2$, $\varphi_\beta(2)=1$, $\beta=1,\cdots,m$ [Bull. Soc. Math. France **81** (1953), 63–75; MR **15**, 313].

L. W. Green (Minneapolis, Minn.)

2099:

Blinčevskiĭ, V. S. Conditions for the absence of uniform and asymptotic stability. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 141–146. (Russian)

Lyapunov's criterion of stability related to the existence of a suitable V function has been shown to be both a necessary and sufficient condition for stability by Malkin, Massera, and Gorbunov. The author in the present paper deals with the analogous Lyapunov negative criteria, and obtains in particular a necessary and sufficient condition for the absence of uniform stability. Consider the system (*) $dx/dt = X(t, x)$, $x = (x_1, \cdots, x_n)$, $X = (X_1, \cdots, X_n)$, $X(t, 0) = 0$, where X is a real continuous function in a cylinder $I(h)$: $\|x\| \leq h$, $t \geq 0$. The zero solution of (*) has no uniform stability if and only if there exists a function $V(t, x)$ continuously differentiable in a cylinder $I(\varepsilon)$, $0 < \varepsilon < h$, bounded in the region $G(V)$ of all points of $I(\varepsilon)$ where $V > 0$, whose total derivative V' is definite positive, and such that $G(V) \cap I(\delta) \neq \emptyset$ for some $\delta > 0$. Also, results are given concerning asymptotic and uniform-asymptotic stability. *L. Cesari* (Baltimore, Md.)

2100:

Chang, H. M. On the invariableness of characteristic exponents under small perturbations. Sci. Record (N.S.) **2** (1958), 351–354.

Conditions on $B(t)$ and $f(t, y)$ are given that the systems $\dot{y} = [A(t) + B(t)]y$ and $\dot{y} = A(t)y + f(t, y)$ have the same characteristic exponents in the sense of Liapunov as the unperturbed system $\dot{x} = A(t)x$. The condition on $B(t)$ is stronger than that given by Bylov [Akad. Nauk SSSR. Prikl. Mat. Meh. **14** (1950), 341–352; MR **12**, 180]. It is sufficient that $B(t) \to 0$ as $t \to \infty$.

J. P. LaSalle (Baltimore, Md.)

2101:

Minc, R. M. Characterization of certain non-rough equilibrium states in three-dimensional space by means of rough equilibrium states of closely similar systems. Dokl. Akad. Nauk SSSR **124** (1959), 1215–1218. (Russian)

The author studies a system of the third order,

$$\dot{x} = P, \quad \dot{y} = Q, \quad \dot{z} = R, \tag{1}$$

where P, Q, R are analytic and zero at the origin, and have no common factor analytic at the origin. Consider the associated analogous system,

$$\dot{x} = P + \bar{P}, \quad \dot{y} = Q + \bar{Q}, \quad \dot{z} = R + \bar{R}. \tag{2}$$

One calls P, Q, R, δ-complements of rank m in a certain closed region g, if $\bar{P}$, $\cdots$, and the absolute values of all partials of order $\leq m$ of $\bar{P}$, $\cdots$ are $< \delta$.

An isolated critical point O of (1) is said to be rough (=structurally stable) if it is the only critical point in some g, and whatever ε there is a δ such that whatever a $g_1 \subset g$ there is an ε topological mapping $g_1 \to g$ sending paths of (1) into those of (2) ((2) of rank one). Theorem: A necessary and sufficient condition for the origin to be a rough critical point is that the characteristic roots of the coefficient matrix of the terms of the first degree of (1) all have non-zero real parts. The study is then made of the case where one of the characteristic roots is zero, sufficient conditions being given in order that the (non-rough) critical point be of one or the other of four certain types in terms of the nature of the rough critical points nearby of suitable systems of type (2). [References: Andronov and Pontrjagin, Dokl. Akad. Nauk SSSR **14** (1937), 247–250; N. A. Gubar', Mat. Sb. (N.S.) **40** (**82**) (1956), 23–56; MR **18**, 805; N. N. Bautin, Akad. Nauk SSSR. Prikl. Mat. Meh. **12** (1949), 613–632; MR **10**, 377; Yu. I. Neĭmark, ibid. **13** (1949), 349–380; MR **11**, 355; R. M. Minc, Dokl. Akad. Nauk SSSR **111** (1956), 535–537; MR **18**, 897.] *S. Lefschetz* (Mexico, D.F.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 2476, 2479.

2102:

Leray, Jean. Problème de Cauchy. I. Uniformisation de la solution du problème linéaire analytique de Cauchy près de la variété qui porte les données de Cauchy. Bull. Soc. Math. France **85** (1957), 389–429.

This article gives details of the author's note in C. R. Acad. Sci. Paris **245** (1957), 1483–1488 [MR **20** #157].

2103:

Harrik, I. Yu. Approximation by functions of special type to functions vanishing, together with their gradient, on the boundary of a region. Mat. Sb. (N.S.) **47** (**89**) (1959), 177–208. (Russian)

Parallel to an earlier work [same Sb. **37** (**79**) (1955), 353–384; MR **17**, 256] this paper develops an approximation theorem for solutions of the boundary value problem: $\Delta^2 u \equiv \sum_{i,j=1}^{m} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} = f(x_1, \cdots, x_m)$, $u = \operatorname{grad} u = 0$ on the boundary Γ. The basic theorem is: If D is a bounded closed region with boundary Γ, k an integer ≥ 2, φ defined in an open set $U \supset D$ and if (1) φ has k continuous derivatives, the kth order derivatives in Lip 1, (2) $\varphi = 0$ on Γ, $\varphi \neq 0$ off Γ, (3) grad $\varphi \neq 0$ on Γ; then there is a sequence of polynomials P_n, deg $P_n \leq n$, such that

$$(*)\quad \|u - \varphi^2 P_n\|_{C_D^r} = O\left\{\frac{\omega^{[k]}_{C(D)}(u; 1/n)}{n^{k-r}}\right\} \quad (r = 0, 1, 2, \cdots, k)$$

for any u continuously differentiable k times in D and such that $u = \operatorname{grad} u = 0$ on Γ.

The definitions of the expressions in (*) are, letting $\Delta_{k_1 \cdots k_m}$ denote $\partial^{k_1 + \cdots + k_m} / \partial x_1^{k_1} \cdots \partial x_m^{k_m}$, as follows:

$$\|f\|_{C_D^r} = \max |\Delta_{k_1 \cdots k_m} f(x)|, \quad 0 \leq k_1 + \cdots + k_m \leq r, \ x \in D;$$

$$\omega^{[k]}_{C(D)}(f; \varepsilon) = \max |\Delta_{k_1 \cdots k_m} f(x) - \Delta_{k_1 \cdots k_m} f(y)|,$$

$$0 \leq k_1 + \cdots + k_m \leq k, \ (x, y) \in \left[\sum_1^m (x_i - y_i)^2 \leq \varepsilon^2\right] \cap D \times D.$$

The estimation procedure involves a lengthy proof which space does not permit us to outline.

B. R. Gelbaum (Minneapolis, Minn.)

2104: [Horváth, John]

Horváth, J. A Basic sets of polynomial solutions for partial differential equations. Proc. Amer. Math. Soc. **9** (1958), 569–575.

The author's main result (as modified by a correction identified at the end of this review) may be paraphrased as follows: Let D be the linear homogeneous partial differential operator of order m, with constant coefficients, $D=\sum_{|M|=m} \alpha_M D^M$, where $M=(m_1, m_2, \cdots, m_n)$, $|M|=m_1+m_2+\cdots+m_n$, and $D^M=\partial^m/\partial x_1^{m_1}\cdots\partial x_n^{m_n}$. Let $M_0=(m_1^0, \cdots, m_n^0)$ be lexicographically minimal among those values of M such that $\alpha_M \neq 0$. In terms of indeterminates $e_1, \cdots, e_n, x_1, \cdots, x_n$ let a be the characteristic polynomial $\sum \alpha_M e^M$ where $e^M=e_1^{m_1}\cdots e_n^{m_n}$, and let $x^j=(x_1e_1+x_2e_2+\cdots+x_ne_n)^j$ for each positive integer j. Let x^j be written (as may be done in just one way) in the form

$$x^j = q_j a + \sum_K Y_{jK} e^K, \tag{1}$$

where q_j is a polynomial in the e's and x's, and each Y_{jK} is a homogeneous polynomial of degree j in the x's, the summation being over all $K=(k_1, \cdots, k_n)$ such that for at least one i the relation $k_i < m_i^0$ is valid. Then the polynomials Y_{jK} are solutions of $D=0$ and form a basis for all those solutions of $D=0$ which are homogeneous polynomials of degree j.

From the relation (1), considered for $j=p$, $j=r$, $j=p+r$, recurrence relations among the Y_{jK} are obtained.

Applications are made to various special cases of D.

This paper unifies, generalizes, and simplifies results and methods of the author, of Miles and Williams, of Protter, of Whittaker, and of Wicht. (Page references are given in the paper.)

{Reviewer's notes: a. The condition of lexicographic minimality of M_0 does not appear in the paper, but is essential, and is supplied above by the reviewer, with the author's approval. b. It is evident that the representation (1) can be effected by an algorithm similar to the division algorithm.} *W. Strodt* (New York, N.Y.)

2105:

Gröbner, Wolfgang. Die Darstellung der Lösungen eines Systems von Differentialgleichungen durch Liesche Reihen. Arch. Math. **9** (1958), 82–93.

The author derives some convergence and combinatorial properties of the Lie series $\sum_{n=0}^{\infty} (t^n/n!)D^n F(z_j)$ and its derivatives, where D is the operator $\sum_{i=1}^{n} \vartheta_i(z_j)\partial/\partial z_i$, and $F(z_j)$ and $\vartheta_i(z_j)$ are functions of the complex variables $z_1, \cdots, z_n$, holomorphic in a suitable region. The functions

$$Z_i = \varphi_i(t; z_j) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n z_i \quad (i = 1, \cdots, n)$$

are defined, and are holomorphic in a neighborhood of $t=0$. It is shown that these functions can be analytically continued by means of Lie series, and that, if F is any function holomorphic in a neighborhood of the point $\{z_i\}$, then $F(Z_j)$ is represented by the Lie series

$$\sum_{n=0}^{\infty} (t^n/n!)D^n F(z_j).$$

The results are applied to exhibit the (unique) solution of the system $\partial Z_i/\partial t=\vartheta_i(Z_j)$, $(Z_i)_{t=0}=z_i$ $(i=1, \cdots, n)$ as a Lie series (namely, $Z_i=\varphi_i(t; z_j)$ $(i=1, \cdots, n)$), and of systems of the form $\partial Z_i/\partial t=\vartheta_i(t; Z_j)$. Characteristics and eigenfunctions are discussed from this viewpoint; some examples are given. *W. J. Coles* (Salt Lake City, Utah)

2106:

Martirosyan, R. M. On the spectrum of the non-selfadjoint operator $-\Delta u+cu$ in three dimensional space. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 85–111. (Russian. Armenian summary)

Let $c(Q)$ be a bounded, complex valued function in $L_2(E)$ where E denotes 3-dimensional space. Author makes a detailed investigation of the spectrum and resolvent of the operator $Tu=-\Delta u+cu$ where Δ is the Laplacian. (The domain Ω of T is the domain of the SA closure of Δ defined initially on the space of those C^∞ functions on E having compact support.) The work constitutes an extension of that of Gel'fand [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6 (52), 183–184; MR **14**, 1091].

Summary of main results: The entire positive real axis belongs to the spectrum of T (Th. 3). Any other points of spectrum T are eigenvalues (Th. 2). Moreover (also Th. 2) the resolvent of T is an integral operator with kernel $H(P, Q; \lambda)$ having the property that, considered as a function either of P or Q separately, $\|H\|_2$ is bounded uniformly in the other variable by a bound depending on λ only. If $\lambda=\xi+i\eta$ and if $\alpha=\|c\|^2/8\pi$ then the entire spectrum of T lies inside the parabola $\eta^2=4\alpha^2(\xi+\alpha^2)$ (Th. 8). There are also several results concerning the distribution of non-positive eigenvalues in the presence of extra conditions on c. Thus (Th. 5) if c is summable (as well as square-summable and bounded) the non-positive point spectrum is bounded. If there exists an $\varepsilon>0$ such that the function $c(Q)\exp(\varepsilon r_Q)$ is both bounded and summable (r_Q is the distance between Q and the origin) then T has only a finite number of non-positive eigenvalues (Ths. 4 and 6). Theorem 7 is a refinement of this last giving a pair of rather complicated conditions insuring that T have no spectrum at all off of the positive real axis.

Proofs, given in full, are mostly of elementary character. An exception is found in Theorem 3 where the argument rests on an interesting extension of the Fredholm alternative due to Kel'dyš that appears to have entered the folklore without getting into print [cf. Gel'fand, loc. cit. supra]. The basic fact throughout is that the operator $-\Delta$ on Ω has for resolvent the integral operator with kernel $\Phi_{\lambda^{1/2}}(r_{PQ})=\exp(i\lambda^{1/2}r_{PQ})/4\pi r_{PQ}$. (Here and always, it is assumed that $\operatorname{Im}(\lambda^{1/2})>0$ and that $\lambda \neq 0$; r_{PQ} is the distance between P and Q.) Suppose now that $\lambda \neq 0$ is an eigenvalue of T and u an eigenfunction: $-\Delta u-\lambda u=-cu$. This is equivalent with

$$u(P)+\int \Phi_{\lambda^{1/2}}(r_{PQ})c(Q)u(Q)dQ = 0$$

which can be iterated by means of a straightforward application of Fubini to yield

$$u(P) = \int K_{\lambda^{1/2}}(P, Q)c(Q)u(Q)dQ, \tag{*}$$

where $K_\lambda(P, Q)$ denotes the kernel

$$K_\lambda(P, Q) = \int \Phi_\lambda(r_{PR})\Phi_\lambda(r_{RQ})c(R)dR.$$

The distribution of eigenvalues is now studied by means of Fredholm theory applied to (*), via a series of quite delicate estimates of $K_\lambda(P, Q)$ (Lemmas 1–5 and 8) the bulk of which may be summarized by saying that K is a bounded and continuous kernel of Hilbert-Smidt type

tending to 0 uniformly on $E \times E$ as $|\lambda| \to \infty$. The investigations of the resolvent kernel H are in a similar vein but too involved to be adequately summarized in a brief space.

A. Brown (Houston, Tex.)

2107a:

Sargsyan, I. S. Asymptotic estimates of the derivatives of a spectral function of the Schrödinger operator on the plane. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 6, 15–30. (Russian. Armenian summary)

2107b:

Sargsyan, I. S. An asymptotic formula for the distribution of the eigenvalues of the Schrödinger operator in a two-dimensional space. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 129–137. (Russian. Armenian summary)

2107c:

Sargsyan, I. S. Expansion and differentiability of expansions in eigenfunctions of the Schrödinger operator in the case of an unboundedly increasing potential. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 193–199. (Russian. Armenian summary)

The author gives, for the two-dimensional case, similar results to those given by B. M. Levitan [e.g. Dokl. Akad. Nauk. SSSR **103** (1955), 191–194; MR **17**, 629].

C. Foiaş (Bucharest)

2108:

Nevanlinna, Rolf. Sur les équations aux dérivées partielles du premier ordre. C. R. Acad. Sci. Paris **247** (1958), 1953–1954.

This is a sketch of a certain approach to the study of the total differential equation $dy = f(x, y)\,dx$, in the context of normed linear spaces. It is presumably intended that the spaces should be complete, though the author does not explicitly say this. The function f is to be defined for $\|x - x_0\| \leq \delta$, $\|y - y_0\| \leq \varepsilon$, where x and y are variables in Banach spaces X, Y, respectively. For each x, y, $f(x, y)$ is to be a bounded linear mapping of X into Y, so that $f(x, y)$ maps dx into dy. One wants a function $y = \varphi(x)$ with $y_0 = \varphi(x_0)$ and Fréchet differential $dy = f(x, \varphi(x))\,dx$.

If X is 1-dimensional and f is subjected to mild restrictions, the existence of a unique solution may be established by classical methods. Hence, regardless of the dimension of X, one may get a solution along a ray emanating from x_0, provided that dx is restricted to be a multiple of $x - x_0$. One may then look upon the integrability condition for the problem as a condition which in some way guarantees that these restricted solutions along line segments in the sphere $\|x - x_0\| \leq \delta$ may be used to construct genuine solutions of the problem.

The present paper confines attention to the case in which $f(x, y)$ is linear in y. Then, if p is a polygonal line from x_1 to x_2 in the sphere $\|x - x_0\| \leq \delta$, we can follow a 'restricted solution' along p, starting from $y = y_1$ at $x = x_1$ and arriving at $x = x_2$ with $y = y_2$. This gives rise to $y_2 = T_p y_1$, where T_p is an operator mapping Y into itself. If the total differential equation is to admit a theorem of existence and uniqueness of solution, then the operators T_p should form a group of automorphisms of Y, and in particular, we should have $T_p = I$ if p is a closed polygon. In a succeeding paper (see the following review) this condition is replaced by a differential condition.

A. E. Taylor (Los Angeles, Calif.)

2109:

Nevanlinna, Rolf. Application d'un principe de E. Goursat dans la théorie des équations aux dérivées partielles du premier ordre. C. R. Acad. Sci. Paris **247** (1958), 2087–2090.

This paper is a sequel to the above. We continue with the notation already explained. The differential equation, assumed linear in y, is written in the form $dy = f(x)\,dx\,y$, and the bilinear operator f is assumed to be continuous when $|x| < r$. The "integrability condition in integral form" ($T_p = I$ for closed polygonal paths) is converted to the form $(T_\gamma - I)/\Delta_\gamma \to 0$ as γ shrinks to x in a fixed 2-dimensional plane, where γ is a triangular path and Δ_γ is its area, measured according to a prescribed convention, using a Euclidean metric in the plane of γ. The method of argument is reminiscent of the proof of Cauchy's theorem by repeated partitions of a triangle.

When f has a continuous Fréchet derivative $f'(x)$, the integrability condition is found to be that which results by setting equal to zero the skewsymmetric part of a certain bilinear form. The reviewer suspects a misprint here, though it may be only a misinterpretation of the notation. The bilinear form in question should apparently be $f'(x)hk - f(x)kf(x)h$ instead of $f'(x)hk - f(x)hf(x)k$. The notation here means that the Fréchet differential of $f(x)hy$ with respect to x (and with $dx = k$) is $f'(x)hky$.

It should be pointed out that an existence theorem for the completely integrable differential equation $dy = F(x, y, dx)$ in Banach spaces was stated and proved by A. D. Michal and V. Elconin [Acta Math. **68** (1937), 71–107]. Their work does not assume F linear in y; the basic hypothesis on F, apart from the integrability condition, is that F has a continuous Fréchet differential with respect to (x, y).

A. E. Taylor (Los Angeles, Calif.)

2110:

Klemola, Tapio. Über lineare partielle Differentialgleichungen erster Ordnung. Ann. Acad. Sci. Fenn. Ser. A. I. no. **260** (1958), 26 pp.

This paper is devoted to a study of the total differential equation $y'(x)\,dx = A(x)\,dx\,y(x)$, where x is a vector in an m-dimensional space R_x, y is a vector in an n-dimensional space R_y, and $A(x)$ denotes a bilinear map from $R_x \times R_y$ into R_y, carrying the pair dx, $y(x)$ into an element of R_y which is supposed to be the Fréchet differential $y'(x)\,dx$. It is assumed that $m > 1$ and that $A(x)$ is continuous in a convex domain V in R_x. The particular interest is in the non-completely-integrable case, and emphasis is laid on the fact that such a light restriction is placed on $A(x)$. The paper follows the general approach of F. and R. Nevanlinna as given in the forthcoming book *Absolute analysis*, to be issued by Springer. See also the papers by R. Nevanlinna reviewed above.

Consider the class of functions $y(x)$ defined in V which satisfy the differential equation in V. Because such a solution is uniquely determined by its value at any selected point, there is an integer p ($0 \leq p \leq n$) independent of x such that the solutions evaluated at x form a p-dimensional space $L^p(x)$ which is a subspace of R_y. If

$p=n$ we have the completely integrable case. The problem of the paper is to attempt to specify a procedure for discovering the value of p. There is a kind of local consistency condition which a vector y_0 must satisfy if it is to be the value at x_0 of a solution $y(x)$ in a convex neighborhood of x_0. This consistency condition is obtained by a limiting process involving displacement of $y(x)$ around a small triangle with one vertex at x_0. The vectors y_0 which satisfy the consistency condition at x_0 form a subspace $K(x_0)$ of R_y. If $A(x)$ is differentiable, the consistency condition is expressible with the aid of a skew-symmetric bilinear form involving the Fréchet derivative of $A(x)$. The sought-for procedure begins with the selection of a maximal family of functions with continuous Fréchet derivatives which, at each x in a neighborhood of x_0, have values which are linearly independent and in $K(x)$. Then a selection process leads, in a finite number of steps, to a basis system of a certain dimension p, from which one can infer the existence of p linearly independent solutions in a neighborhood x_0; it may be, of course, that $p=0$. The details of the argument are too long to be described here.

A. E. Taylor (Los Angeles, Calif.)

2111:

Yoshimatsu, Senjiro. Sur les équations aux dérivées partielles du premier ordre. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **4** (1955), 41–44.

A slight modification of a theorem by M. Nagumo [Jap. J. Math. **18** (1942), 41–47; MR **7**, 382] is proposed. This reviewer is unable to follow the author's argument.

E. H. Rothe (Ann Arbor, Mich.)

2112:

Višik, M. I.; and Lyusternik, L. A. The asymptotic behaviour of the solutions of differential equations containing large and rapidly changing coefficients. Dokl. Akad. Nauk SSSR **125** (1959), 247–250. (Russian)

The authors have studied [same Dokl. **113** (1957), 734–737; **119** (1958), 636–639; **120** (1958), 13–16; Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5, 3–122; MR **19**, 861; **20** #4696; **21** #219; **20** #2539] the asymptotic behavior of solutions of partial differential equations of any order having large coefficients of the lower derivatives. In this paper they illustrate their methods and results by considering some special examples of second order elliptic equations.

A. Friedman (Minneapolis, Minn.)

2113:

Dymkov, S. S. The first boundary problem for quasi-linear elliptical equations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 220–222. (Russian)

Let Ω be a bounded domain in E^n with smooth boundary S. It is proved that the boundary value problem $u=0$, $x\in S$,

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,u)\frac{\partial u}{\partial x_i} + a(x,u) = 0,$$

$(x=(x_1, \cdots, x_n)\in\Omega)$, possesses a solution in the Hölder class $E_{\alpha,2}$ under the following assumptions:

$$(1) \qquad \partial a(x,u)/\partial u \geq \beta > 0$$

for $x\in\Omega$ and all values of u; (2) the functions a_{ij}, a_i, and a and their first derivatives with respect to all arguments are bounded for $x\in\overline{\Omega}$ and $|u|\leq C_1=\beta^{-1}\max_{x\in\overline{\Omega}}|a(x,0)|$; (3) for some $\alpha>0$ and arbitrary real ξ_i,

$$\sum_{i,j=1}^{n} a_{ij}(x,u)\xi_i\xi_j \geq \alpha\sum_{i=1}^{n}\xi_i^2, \quad \max\left|\frac{\partial a_{ij}}{\partial u}\right| \leq \frac{\alpha e\sqrt{3}}{12nC_1}$$

for $x\in\overline{\Omega}$, $|u|\leq C_1$. A related existence theorem, in which (3) is replaced by a different condition, and a uniqueness theorem are also given. The proofs are based, in part, on a paper by O. A. Ladyženskaya [same Dokl. **107** (1956), 636–639; MR **17**, 1214].

P. Henrici (Los Angeles, Calif.)

2114:

Simoda, Seturo. Identité des solutions de l'équation $\Delta u = F(x, u, \text{grad } u)$ au sens du laplacien généralisé et au sens du laplacien ordinaire. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **4** (1955), 35–40.

Il lavoro è dedicato all'identità tra soluzioni "deboli" e soluzioni "ordinarie" dell'equazione

$$(I) \qquad \Delta u = F(x, u, \partial u/\partial x_1, \cdots, \partial u/\partial x_n).$$

Viene introdotto assiomaticamente un concetto di laplaciano generalizzato nel seguente modo. Sia Ω un aperto di R^n; supponiamo che ad ogni funzione $u\in C(\Omega)$ e ad ogni x di Ω siano associati due valori (eventualmente $-\infty$ e $+\infty$) $\overline{\Delta}u(x)$ e $\underline{\Delta}u(x)$ tali che (1) $\overline{\Delta}u(x)\leq\underline{\Delta}u(x)$; (2) $\overline{\Delta}(-u(x))=-\underline{\Delta}u(x)$; (3) $\overline{\Delta}(u+v)(x)\geq\overline{\Delta}u(x)+\overline{\Delta}v(x)$, $\underline{\Delta}(u+v)(x)\leq\underline{\Delta}u(x)+\underline{\Delta}v(x)$; (4) $\overline{\Delta}u(x_0)\geq 0$ [$\underline{\Delta}u(x_0)\leq 0$] se x_0 è un punto di minimo [massimo] per u; (5) se u è due volte differenziabile con continuità in un intorno di x_0 allora è $\overline{\Delta}u(x_0)=\underline{\Delta}u(x_0)=\sum_{i=1}^{n}\partial^2u(x_0)/\partial x_i^2$; (6) se $u(x)=\int_\Omega f(\xi)e(x,\xi)d\xi$, essendo $e(x,\xi)$ la soluzione fondamentale dell'equazione di Laplace e $f(x)\in C(\Omega)$ e a supporto compatto in Ω, allora è $\overline{\Delta}u(x)=\underline{\Delta}u(x)=f(x)$ in Ω.

Si dice che u possiede laplaciano generalizzato in x se gli operatori $\overline{\Delta}u(x)$ e $\underline{\Delta}u(x)$ hanno valore finito in x e coincidono, e si pone $\Delta u(x)=\overline{\Delta}u(x)=\underline{\Delta}u(x)$.

Il risultato principale è il seguente: se $F(x, u, p_1, \cdots, p_n)$ è continua e tale che $F(x, u(x), p_1(x), \cdots, p_n(x))$ è localmente hölderiana in Ω quando lo siano le $n+1$ funzioni $u(x), p_1(x), \cdots, p_n(x)$, allora per ogni funzione $u\in C^1(\Omega)$ e soddisfacente all'equazione (I), risulta anche $u\in C^2(\Omega)$ e dunque la (I) è verificata in senso ordinario.

E. Magenes (Genoa)

2115:

Yosida, Kosaku. Integration of the wave equation by the theory of semi-groups. Sûgaku **8** (1956/57), 65–71. (Japanese)

A refined, more expository version of the author's paper in J. Fac. Sci. Univ. Tokyo Sect. I **8** (1957), 463–466 [MR **19**, 423]. The method employed here is more of distribution theory.

H. Yamabe (Minneapolis, Minn.)

2116:

Chu, Pei-chang; and Ling, Ling. A maximum principle of the singular third problem for Euler-Poisson-Darboux's equation. Sci. Record (N.S.) **1** (1957), 373–374.

La théorème du maximum pour le troisième problème aux limites singulier, avec des données nulles sur la caractéristique, relatif à une équation d'Euler-Poisson, établi pour la première fois par P. Germain et R. Bader

[O.N.E.R.A. Publ. no. **54** (1952); MR **14**, 654], est démontré par une méthode reposant sur l'expression explicite de la solution de ce problème. La méthode proposée ne diffère donc pas de celle de P. Germain et R. Bader, de façon essentielle. Rappelons que S. Agmon, L. Nirenberg et M. H. Protter ont établi un théorème du maximum dans des conditions beaucoup plus générales [Comm. Pure Appl. Math. **6** (1953), 455–470; MR **15**, 432].

P. Germain (Paris)

2117:

★**Weinstein, A. Sur une classe d'équations aux dérivées partielles singulières.** La théorie des équations aux dérivées partielles. Nancy, 9–15 Avril 1956, pp. 179–186. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

Dans cette conférence je me propose de donner une revue de quelques résultats récents obtenus dans la théorie des équations du type

$$\frac{\partial^2 u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}+Xu=0,$$

$u=u(x_1, x_1, \cdots, x_m, y)=u(x, y)$; $x=(x_1, x_2, \cdots, x_m)$, $m=1, 2, \cdots$; $-\infty<k<\infty$. X désigne un opérateur de second ordre qui ne dépend que de x. *De l'introduction*

2118:

Bicadze, A. V. Incorrectness of Dirichlet's problem for the mixed type of equations in mixed regions. Dokl. Akad. Nauk SSSR **122** (1958), 167–170. (Russian)

Man kennt aus der Theorie der hyperbolischen wie auch aus der Theorie der elliptischen Differentialgleichungen das Problem der "sachgemäß" bzw. "unsachgemäß" gestellten Randwertaufgaben.

Verfasser behandelt in diesem Zusammenhang die auch von M. A. Lawrentjew studierte Differentialgleichung gemischten Typs $U_{xx}+\operatorname{sgn} y\cdot U_{yy}=0$. Der Definitionsbereich D dieser Differentialgleichung zerfällt in einen hyperbolischen und einen elliptischen Teil. Für den hyperbolischen Teil kann man das Dirichletsche Problem stellen und fragen, inwiefern die Lösung des Randwertproblems im hyperbolischen Teil die Lösung für den elliptischen Teil beeinflußt. Verfasser gewinnt die Bedingungen für eine eineindeutige Lösung des gemischten Problems, d.h. für die Existenz einer Lösung $U(x, y)$ der Lawrentjewschen Differentialgleichung, innerhalb D für $y\neq 0$, $y\neq x-x_0$, $y\neq x_0-x$, welche im abgeschlossenen Bereich $\bar{D}$ stetig ist mit stetigen partiellen Ableitungen U_x, U_y (abgesehen vom Verhalten in singulären Punkten, etwa längs der Charakteristiken $y=x-x_0$ und $y=x_0-x$, in welchen diese partiellen Ableitungen über alle Grenzen wachsen können).

Unter vereinfachenden Annahmen wird für die Lösung $U(x, y)$ eine explizite Darstellung angegeben. Weiterhin werden die Normen der auftretenden singulären Integrale abgeschätzt und Zusammenhänge mit Fredholmschen Integralgleichungen besprochen. *M. Pinl* (Cologne)

2119:

Mizohata, Sigeru. Unicité dans le problème de Cauchy pour quelques équations différentielles elliptiques. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 121–128.

Let P be an elliptic operator with real, constant coefficients, homogeneous, of order $2m$. The uniqueness of the Cauchy problem is established for equations of the form $(P+Q)u=0$ when Q is a linear operator of order less than or equal to m. Under certain additional hypotheses on the multiplicity of the characteristic roots of P the order of the operator Q may be higher than m. The proof employs L_2 estimates obtained by Fourier transforms [see Nirenberg, Comm. Pure Appl. Math. **10** (1957), 85–105; MR **19**, 147]. *M. H. Protter* (Berkeley, Calif.)

2120:

Danyluk, I. I. Über das System von Differentialgleichungen erster Ordnung vom elliptischen Typus auf Riemannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I. no. 251/2 (1958), 7 pp.

The author generalizes I. N. Vekua's theory of systems of first order partial differential equations of elliptic type [Math. Sb. (N.S.) **31** (**73**) (1952), 217–314; MR **15**, 230] to the case of finite Riemann surfaces. Let $\overline{A(z)}$ and $\overline{B(z)}$ be matrices and $\overline{\phi(z)}$ a vector whose elements transform like differentials of the Riemann surface considered under a change of uniformizer. Consider the differential system $\frac{\partial f}{\partial \bar{z}}=A(z)f(z)+B(z)\overline{f(z)}+\phi(z)$ for the function vector $f(z)=(f_1(z), f_2(z), \cdots, f_n(z))$. Following Vekua's procedure this system is transformed into an equivalent inhomogeneous integral equations system. As the kernel appears the expression $h(\zeta, z)$, which is a function in z and a differential in ζ, and which has a simple pole for $z=\zeta$ with residue 1. If the matrices A and B are triangular, the system has a unique solution which is obtained by successive approximation. An explicit expression is given for the solution vector in terms of its values on the boundary of the finite Riemann surface. *M. Schiffer* (Stanford, Calif.)

2121:

Vinogradov, V. S. On a boundary value problem for linear elliptic systems of first order in the plane. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 1059–1062. (Russian)

In this note, the author treats the boundary value problem

$$\operatorname{Re}\{(\alpha-i\beta)w\}|_\Gamma=0 \tag{1}$$

for functions w having generalized L_p $(p>2)$ derivatives in the sense of Sobolev, which satisfy the differential equations

$$\frac{\partial w}{\partial \bar{z}}+\mu_1(z)\frac{\partial w}{\partial z}+\mu_2(z)\overline{\frac{\partial w}{\partial z}}+a(z)w(z)+b(z)\overline{w(z)}=g(z) \tag{2}$$

in a simply connected domain D bounded by a curve Γ with continuous curvature. Here $\alpha(t)$, $\beta(t)$ are given Hölder continuous functions on Γ, μ_1, μ_2 are bounded measurable functions satisfying $|\mu_1|+|\mu_2|\leq g<1$ in D, and a, b, g are of class $L_p(D)$, $p>2$. When μ_1 and μ_2 are differentiable, the problem may be reduced to a similar boundary value problem for solutions of $w_z+Aw+B\bar{w}=G$ [cf. Bers and Nirenberg, Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 111–140, Edizioni Cremonese, Rome, 1955; MR **17**, 974]. The latter has essentially been treated by I. N. Vekua [Mat. Sb. (N.S.) **31** (**73**) (1952), 217–314; MR **15**, 230].

The general problem may be reduced to the special boundary value problem

$$\text{Re}\,\{t^{-n}w(t)\}|_\Gamma = 0 \tag{3}$$

($n=(2\pi)^{-1}$ [arg $(\alpha+i\beta)]_\Gamma$), for functions w satisfying an equation of the form (2) in the unit disc, $|z|<1$. {Reviewer's comment: In order to be certain that the coefficients of (2) remain in L_p, $p>2$, it is necessary to assume that the Hölder exponents of α, β exceed $\frac{1}{2}$.} Using integral representations for functions w having L_p derivatives and satisfying (3), one is led, through (2), to a two-dimensional singular integral equation, which, when p is sufficiently close to 2, is equivalent to a Fredholm equation. A representation theorem of Bers and Nirenberg [loc. cit.] then allows one to conclude that the homogeneous singular integral equation has only the trivial solution. {At the beginning of line 11, p. 1061, $w(z)|_\Gamma$ should be replaced by $iz^{-n}w(z)|_\Gamma$.} The author's reasoning in the concluding parts of this proof is incorrect. The results remain intact, however, since we can replace the author's argument by one given by Nitsche [Math. Nachr. **14** (1955), 75–127; MR **19**, 554]. The following results are obtained. When $n\geq 0$, the inhomogeneous problem (2), (3) always has a solution, while the homogeneous problem ($g\equiv 0$) has $2n+1$ linearly independent solutions. When $n<0$, the homogeneous problem has only the trivial solution, while the inhomogeneous problem has a solution if and only if the transform of g, by the inverse of the singular integral operator which occurs in the singular integral equation, satisfies a set of $-2n-1$ integral conditions.

W. Koppelman (Cambridge, Mass.)

2122:

Friedman, Avner. Linear partial differential systems with an additional differential equation at one point. J. Math. Mech. **7** (1958), 173–190.

It is shown that all solutions of a linear differential system of elliptic or parabolic type and of order m cannot satisfy an additional linear partial differential equation of order m at a fixed point unless the new equation reduces at this point to a linear combination of the original equations. The coefficients of the ith order terms ($i\leq m$) are assumed to be i times continuously differentiable. In addition, it is shown that one can solve for the coefficients of an elliptic or hyperbolic linear differential equation system in terms of its solutions.

H. C. Kranzer (New York, N.Y.)

2123:

Pogorzelski, W. Étude de la matrice des solutions fondamentales du système parabolique d'équations aux dérivées partielles. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 79–83.

The matrix of fundamental solutions is constructed for a parabolic system of N partial differential equations

$$\partial u_i/\partial t = \Psi^{(0)}_{X,t}(u_1, \cdots, u_N) \quad (i = 1, \cdots, N)$$

of even order M, where the coefficients are complex functions of (X, t) with X a point in n-dimensional Euclidean space ($n\geq 2$), and $0\leq t\leq T$. The coefficients of the derivatives of highest order are assumed to satisfy Hölder conditions in X and t; the lower order coefficients satisfy Hölder conditions in X, but only continuity in t. No proofs are given here, but they have been published [Ricerche Mat. **7** (1958), 153–185].

J. Elliott (New York, N.Y.)

2124:

★John, Fritz. Prolongement et réflexion des solutions des équations aux dérivées partielles. La théorie des équations aux dérivées partielles. Nancy, 9–15 Avril, 1956, pp. 103–109. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

L'Auteur considère le problème du prolongement, à travers le plan $t=0$, des solutions des équations du type

$$P(\xi_i, \tau)u(x_i, t) = 0,$$

où x_i, $1\leq i\leq u$, et t sont les variables indépendantes, et P une forme de degré N en $\xi_i=\partial/\partial x_i$ et $\tau=\partial/\partial t$; avec les conditions aux limites $\tau^k u(x_i, t)=0$ ($k=0, 1, \cdots, s-1$) sur le plan $t=0$.

Le théorème général suivant est établi: une condition nécessaire pour que toute solution C_∞ puisse être prolongée, est que pour chaque vecteur réel η_i l'équation $\underline{P}(\eta_i, \lambda)=0$ ou bien ait seulement des racines réelles λ, ou bien ait au moins $(N-s)$ racines à partie imaginaire positive, et $(N-s)$ racines à partie imaginaire négative. Il en résulte un corollaire simple pour le cas $s<\frac{1}{2}N$.

Dans le cas où $N=2m$, $s=m$ l'auteur définit un principe de réflexion, et il établit le théorème ci-après.

Pour qu'il y ait un principe de réflexion, il est nécessaire et suffisant que $P(\xi_i, \tau)$ soit de l'un des 3 types:

(I) $$P = \tau^{2m};$$

(II) $$\underline{P} = \prod_{k=1}^{2m} [\tau - c_k L(\xi_i)],$$

où L est une forme linéaire, indépendante de k, et les c_k des constantes réelles, $c_1, \cdots, c_m>0$ et $c_{m+1}, \cdots, c_{2m}<0$;

(III) $$\underline{P} = \prod_{k=1}^{m} [\tau^2 - c_k{}^2 Q(\xi_i)],$$

où Q est une forme quadratique indépendante de k et les c_k des constantes réelles positives.

L'équation $\Delta^p u=0$ est un cas spécial de (III) pour lequel $u(x_i, t)$ s'obtient comme résultat de l'application d'un opérateur différentiel à $u(x_i, -t)$.

R. Gerber (Grenoble)

POTENTIAL THEORY

See also 2056, 2114.

2125:

Avanissian, Vazguen. Mesures dépendant harmoniquement d'un ou de deux groupes de variables. C. R. Acad. Sci. Paris **247** (1958), 2278–2281.

L'A. considère une mesure générale de Radon $\mu(f)$, ou une forme linéaire sur un sous-espace des f, dépendant d'un paramètre y dans un domaine D_y de R^p. Il l'étudie lorsqu'elle est harmonique de y et que sa norme reste bornée quand y décrit un compact de D_y. Il introduit

aussi un second paramètre z dans R^q, (y, z) variant dans un domaine ou ouvert D de R^{p+q}, en particulier $D_x \times D_y$, avec la propriété précédente quand on fixe y ou z et une limitation de la norme pour (y, z) décrivant un compact de D. On donne des théorèmes de prolongement (de la mesure), en particulier à partir de $D_x \times D_y$ diminué de la variété W: $(y=0)\times D_z$ $(0 \in D_y)$ sur laquelle on prolonge. On passe à l'étude d'une fonction $V(y, z)$ bornée localement dans $D = D_y \times D_z$, sousharmonique en y pour z fixé et harmonique en z pour y fixé.—Théorèmes de prolongement à partir de $D - W$, avec des hypothèses au voisinage de W.

M. Brelot (Paris)

2126:

Milicer-Grużewska, H. Propriété limite du potentiel généralisé de simple couche relativement à l'équation parabolique normale. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 495–499.

As shown by Pogorzelski [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 9–13; Ricerche Mat. **5** (1956), 25–57; MR **18**, 47], the generalized potential of a simple layer relative to the linear parabolic equation

$$(1) \qquad \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} b_\alpha \frac{\partial u}{\partial x_\alpha} + cu - \frac{\partial u}{\partial t} = 0$$

is the integral

$$U(A, t) = \int_0^t \iint_S \Gamma(A, t; Q, \tau)\phi(Q, \tau)dQd\tau$$

where Γ is the fundamental solution of (1), ϕ is the density and S is a closed surface bounding an euclidean n-space Ω.

Under assumptions similar to but weaker than those of Pogorzelski [Ann. Polon. Math. **3** (1957), 247–284; MR **19**, 282] and Krzyżański [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 247–251; MR **18**, 47] (no differentiability conditions are imposed on the coefficients, but they are assumed to have uniform limits as $t \to \infty$), the author states that the limit as $t \to \infty$ of the potential $U(A, t)$ of the single layer of density $\phi(Q, t)$ relative to equation (1) is equal to the potential of the limit density $\bar{\phi}(Q) = \lim_{t\to\infty} \phi(Q, t)$ relative to the elliptic equation

$$\sum \bar{a}_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum \bar{b}_\alpha \frac{\partial u}{\partial x_\alpha} + \bar{c}u = 0,$$

where the bar denotes the uniform limit of the corresponding functions in equation (1). The proof appears in Ann. Scuola Norm. Sup. Pisa **12** (1958), 359–396.

C. G. Maple (Ames, Iowa)

2127:

Siciak, J. Sur la distribution des points extrémaux dans les ensembles plans. Ann. Polon. Math. **4** (1958), 214–219.

Let E be a closed and bounded set of points in the plane and let $f(z)$ be a real continuous function defined on E. Adopting the terminology introduced by F. Leja [same Ann. **3** (1957), 319–342; MR **19**, 645], the author considers a system of extremal points

$$(1) \qquad \eta^{(n)} = \{\eta_0^{(n)}, \eta_1^{(n)}, \cdots, \eta_n^{(n)}\}$$

of rank n of E, with respect to the function $\omega(z, \zeta) = |z-\zeta|e^{-f(z)-f(\zeta)}$ and the set E^* of limit points of the triangular sequence (1). Since the system (1) of extremal points need not, in general, be unique, it is not a priori evident that E^* will be independent of the particular sequence (1) chosen. The author gives a sufficient condition under which the set E^* is independent of the choice of the extremal system (1). Again referring to Leja's terminology, it is shown that if the "écart" $v(E, f)$ is positive and if the set E possesses property W (called P in this paper) at every one of its points, then the transfinite diameter of $K(z_0, r) \cap E^*$, where z_0 is any point of E^* and $K(z_0, r)$ is the disk $|z-z_0| \leq r$, is positive for every $r > 0$ and the set E^* is independent of the choice of the extremal sequence (1). A complete characterization of E^* is obtained in the case that E satisfies the above conditions and moreover is the frontier of a region which contains the point at infinity in its interior. An example shows that if E does not satisfy the latter condition, the above characterization of E^* need no longer be valid.

W. Seidel (Notre Dame, Ind.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 2077, 2193.

2128:

Spitzbart, A.; and Macon, N. Numerical differentiation formulas. Amer. Math. Monthly **64** (1957), 721–723.

If $f(x)$ is the polynomial of degree n passing through $n+1$ equidistant points (x_i, y_i), then

$$h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi}{}^k y_i + O(h^{n+1}),$$

where

$$A_{mi}{}^k = \sum_{j=k}^{n} (-1)^{i+j}\binom{j}{i} C_{mj}{}^k / j!,$$

$$C_{mj}{}^k = \sum_{r=k}^{j} r^{(k)} S_j{}^r m^{r-k},$$

$r^{(k)}$ is the factorial polynomial of degree k, $S_j{}^r$ are the Stirling numbers, and $x = x_0 + mh$.

In a recent paper [R. T. Gregory, same **64** (1957), 79–89; MR **18**, 767] expressions for the coefficients $A_{mi}{}^k$ were obtained as the inverses of certain Vandermonde matrices. This connection enables the author to use the results above to obtain inverses of Vandermonde matrices [see the following review]. *D. C. Gilles* (Glasgow)

2129:

Macon, N.; and Spitzbart, A. Inverses of Vandermonde matrices. Amer. Math. Monthly **65** (1958), 95–100.

The results of a paper by the same authors [reviewed above] enable the inverses of Vandermonde matrices to be determined.

The matrices considered are those whose elements $x_0, \cdots, x_r, \cdots, x_n$ are (i) equally spaced at unit intervals, $x_r = -m + r$, (ii) equally spaced at interval h, $x_r = x_0 + rh$, and (iii) any distinct numbers. *D. C. Gilles* (Glasgow)

SEQUENCES, SERIES, SUMMABILITY

See also 1943.

2130:

Šalát, T. Über einige Probleme aus der Theorie der unendlichen Reihen. Acta Fac. Nat. Univ. Comenian. **3**, 29–39 (1958). (Slovak. Russian and German summaries)

Verfasser betrachtet Reihen komplexer Zahlen $\sum_{n=1}^{\infty} \varepsilon_n a_n$, die aus einer festen Reihe $\sum a_n$ durch Faktoren $\varepsilon_n \in M_n$ entstehen, wo jedes M_n eine nichtleere endliche Menge ist. Der Abstand zweier solcher Reihen (mit Faktoren ε_n und ε_n') wird definiert als $1/l$, wo l die kleinste Zahl mit $\varepsilon_l \neq \varepsilon_l'$ ist. Der so entstehende metrische Raum ist vollständig und kompakt und hat die Dimension 0. Gilt $\sum K_l|a_l| < \infty$ (wo $K_l = \mathrm{Max}_{z \in M_l}|z|$), so ist der Operator, der jeder Reihe ihre Summe zuordnet, stetig und die Menge aller Summen kompakt. Es wird nun vorausgesetzt: $\sum a_n = \infty$, $a_n > 0$, $a_n \to 0$, $M_l = \{1, -1\}$. Dann ist die Menge der Reihen $\sum \varepsilon_n a_n$, die zwischen $\pm\infty$ oszillieren, in jedem Punkt des Raumes von II. Kategorie. Die Menge der Reihen, die zwischen κ und μ oszillieren (wo $-\infty \leq \kappa \leq \mu \leq +\infty$), hat die Mächtigkeit des Kontinuums. Außerdem beweist Verfasser einen ähnlichen Satz von H. M. Sengupta [Proc. Amer. Math. Soc. **1** (1950), 71–75; MR **11**, 346] über Umordnung von Reihen. Die Ergebnisse sind teilweise schon in früheren Veröffentlichungen des Autors enthalten [Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4** (1954), 203–211; **5** (1955), 94–100; **7** (1957), 139–142; MR **16**, 1099; **17**, 1436, 728; **20** #4723].

K. Zeller (Tübingen)

2131:

Tandori, Károly. Summation of orthogonal series. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 397–402. (Hungarian)

The author's theorems deal with series of the form $\sum_0^{\infty} a_n\phi_n(x)$ where the $\phi_n(x)$ are an orthonormal set, and assert the existence of counter-examples to prove the sharpness of various known results. In the case of Rademacher's result [Math. Ann. **87** (1922), 112–138] that if $\sum a_n^2 < \infty$ then $\sum_{n=0}^{N} a_n\phi_n(x) = o(\log N)$ almost everywhere, this bound remains best possible even if it is also assumed that $\sum_0^{\infty} a_n\phi_n(x)$ is summable (C, α) for any $\alpha > 0$ almost everywhere. Let next $\sum_0^{\infty} a_n^2 < \infty$, $n^{1/2}a_n \geq (n+1)^{1/2}a_{n+1}$; then the criterion $\sum_0^{\infty} (a_n \log\log n)^2 < \infty$ is necessary and sufficient for $\sum_0^{\infty} a_n\phi_n(x)$ to be summable (C, α) almost everywhere for any $\alpha > 0$, and for all sets $\phi_n(x)$ [cf. D. Menchoff, Fund. Math. **8** (1926), 56–108]. The final result shows that the result of G. Alexits [Acta Math. Acad. Sci. Hungar. **7** (1956), 5–9; MR **18**, 124] cannot be improved. Detailed proofs appear in Acta Sci. Math. Szeged **18** (1957), 149–168, 169–178; see also the first part of the same work [ibid. **18** (1957), 57–130; MR **19**, 851].

F. V. Atkinson (Canberra)

2132:

Brudno, A. L. An example of two Toeplitz matrices not boundedly contradictory and not boundedly coverable. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 309–320. (Russian)

The author shows the existence of two Toeplitz matrices A and B such that: (1) no bounded sequence is summable by the matrices A and B to different limits, (2) no Toeplitz matrix is able to sum all bounded sequences summable by at least one of the matrices A and B. The example of the matrices A and B is based on geometrical considerations.

M. Fiedler (Prague)

2133:

Szekeres, G.; and Jakimovski, A. (C, ∞) and (H, ∞) methods of summation. Pacific J. Math. **8** (1958), 867–886.

Let $T^{(r)}$ denote the Cesàro or Hölder kernel transformation of order r so that, for an appropriate kernel having values $a^{(r)}(t, x)$, the $T^{(r)}$ transform of a given function f has values

$$\sigma^{(r)}(t) = \int_0^{\infty} a^{(r)}(t, x)f(x)dx,$$

when $0 < t < \infty$ and, where t_0 may be 0 or ∞, the given f is $T^{(r)}$ limitable to L as $t \to t_0$ if $\sigma^{(r)}(t) \to L$ as $t \to t_0$. In contrast to other investigations in which f is said to be $T^{(\infty)}$ limitable to L if

$$\lim_{r\to\infty} \limsup_{t\to t_0} |\sigma^{(r)}(t) - L| = 0,$$

the present paper defines f to be $T^{(\infty)}$ limitable to L if

$$\lim_{t\to t_0} \lim_{r\to\infty} \sigma^{(r)}(t) = L.$$

That $T^{(r)} \subset T^{(\infty)}$ when $t_0 = 0$ (but not when $t_0 = \infty$) is a consequence of the following theorem. If $\lim_{t\to\infty} \sigma^{(r)}(t) = L$ for some $r \geq 0$, then $\lim_{r\to\infty} \sigma^{(r)}(t) = L$ for each $t > 0$. Let $C^{*(r)}$ and $H^{*(r)}$ denote the modified Cesàro and Hölder transformation which is based upon the transform $x \int_x^{\infty} t^{-2}f(t)dt$ in the same way that the ordinary Cesàro and Hölder transformations $C^{(r)}$ and $H^{(r)}$ are based upon the transform $x^{-1} \int_0^x f(t)dt$. Results involving $C^{*(\infty)}$ and $H^{*(\infty)}$ are analogous to those involving $C^{(\infty)}$ and $H^{(\infty)}$, but the roles of $t_0 = 0$ and $t_0 = \infty$ are reversed.

R. P. Agnew (Ithaca, N.Y.)

2134:

Wilansky, Albert; and Zeller, Karl. Banach algebra and summability. Illinois J. Math. **2** (1958), 378–385; erratum **3** (1959), 468.

Consider the set of matrices of finite norm $\|A\|$. Define $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$. Γ is the set of conservative matrices and Δ is the set of conservative triangular matrices. The radical is the set of matrices A such that, for every B, $I - AB$ has an inverse.

The authors prove that for both Γ and Δ a matrix is in the radical if and only if its diagonal elements are 0 and $\sum_{k=1}^{\infty} |a_{nk}|$ is uniformly convergent. Several applications are given.

Many other theorems are proved; e.g., if x is a bounded divergent sequence and y a bounded sequence, there exists $A \in \Delta$ such that all but a finite number of terms of $y - Ax$ are 0, but there exist unbounded sequences x and y such that $y - Ax$ is divergent for every $A \in \Delta$.

P. Erdös (Boulder, Colo.)

APPROXIMATIONS AND EXPANSIONS

See also 2079, 2103, 2131, 2151, 2341.

2135:

Balázs, J.; and Turán, P. Notes on interpolation. IV. Inequalities. Acta Math. Acad. Sci. Hungar. **9** (1958), 243–258.

As an application of their earlier results on interpolation [same Acta **8** (1957), 201–215; **9** (1958), 195–214; MR **19**, 544; **20** #4725; for part I see Surányi and Turán, ibid. **6** (1955), 67–80; MR **17**, 148] the authors obtain the following interesting estimates. Let x_ν denote the zeros of $(1-x^2)P'_{n-1}(x)$. If a polynomial $f(x)$ of degree $2n-1$ satisfies the conditions $|f(x_\nu)| \leq A$, $|f''(x_\nu)| \leq B$, then we have for $-1 \leq x \leq 1$ the following inequality: $|f(x)| \leq \pi^6 nA + \pi^5 n^{-1}B$. Under the same condition as before we have $|f'(x)| \leq \pi^8 n^{5/2}A + \pi^5 n^{1/2}B$. These bounds are the best possible so far as the order of magnitude in n is concerned. The second inequality can be sharpened to an essential extent if we seek the bound of $f'(x)$ in an interval $-1+\varepsilon \leq x \leq 1-\varepsilon$. *G. Szegö* (Stanford, Calif.)

2136:

Egerváry, E.; and Turán, P. Notes on interpolation. V. On the stability of interpolation. Acta Math. Acad. Sci. Hungar. **9** (1958), 259–267.

Let x_ν $(\nu=1, 2, \cdots, n)$ be given distinct points of interpolation in $[-1, 1]$. We consider all polynomials $r_\mu(x)$ satisfying the conditions $r_\mu(x_\nu)=\delta_{\mu\nu}$. Moreover we assume that $0 \leq r_\mu(x) \leq 1$, $-1 \leq x \leq 1$. The authors determine the minimum of $\sum_{\mu=1}^{n}$ (degree of $r_\mu(x)$); it is attained when the x_ν coincide with the zeros ξ_ν of $P_{n-2}(x)$ and ± 1. The resulting interpolation polynomials

$$\sum_{\nu=1}^{n-2} f(\xi_\nu)\frac{1-x^2}{1-\xi_\nu^2}\left(\frac{P_{n-2}(x)}{P'_{n-2}(\xi_\nu)(x-x_\nu)}\right)^2 + f(1)\frac{1+x}{2}(P_{n-2}(x))^2 + f(-1)\frac{1-x}{2}(P_{n-2}(x))^2$$

converge uniformly to $f(x)$ if $f(x)$ is continuous in $[-1, 1]$. The following remarkable identity is obtained (ξ_ν are now the zeros of $P_n(x)$):

$$1-(P_n(x))^2 = \sum_{\nu=1}^{n}\frac{1-x^2}{1-\xi_\nu^2}\left(\frac{P_n(x)}{P_n'(\xi_\nu)(x-\xi_\nu)}\right)^2.$$

It puts in evidence the classical inequality $|P_n(x)| \leq 1$, $-1 \leq x \leq 1$. *G. Szegö* (Stanford, Calif.)

2137:

Freud, G. Bemerkung über die Konvergenz eines Interpolationsverfahrens von P. Turán. Acta Math. Acad. Sci. Hungar. **9** (1958), 337–341.

Let n be even and let us denote by x_ν the zeros of $(1-x^2)P'_{n-1}(x)$; $(x_1=1, x_n=-1)$. The polynomials of degree $2n-1$

$$R_n(x; f) = \sum_{\nu=1}^{n}[f(x_\nu)r_{\nu n}(x)+\beta_{\nu n}\rho_{\nu n}(x)]$$

are defined by the conditions $R_n(x_\nu)=f(x_\nu)$, $R_n''(x_\nu)=\beta_{\nu n}$. In joint papers of Balázs and Turán, and Surányi and Turán [#2135 above and references therein], the uniform convergence of these polynomials was discussed. In the present paper the conditions on $f(x)$ are sharpened in the following manner: $f(x)$ is continuous and

$$|f(x+h)-2f(x)+f(x-h)| \leq \varepsilon(h),$$

$(x-h, x+h) \in [-1, 1]$, where $\varepsilon(h)/h \to 0$. Moreover,

$$\max((1-x_\nu)^{1/2}|\beta_{\nu n}|\cdot n^{-1}, |\beta_{1n}|n^{-2}, |\beta_{nn}|n^{-2}),$$

where $2 \leq \nu \leq n-1$, must tend to zero as $n \to \infty$. *G. Szegö* (Stanford, Calif.)

2138:

Saxena, R. B.; and Sharma, A. On some interpolatory properties of Legendre polynomials. Acta Math. Acad. Sci. Hungar. **9** (1958), 345–358.

The authors deal with the following interpolation of "type" (0, 1, 3). Let x_ν be a system of distinct values in $[-1, 1]$; we seek a polynomial $f(x)$ of degree $3n-1$ such that

$$f(x_\nu)=y_\nu, \quad f'(x_\nu)=y_\nu', \quad f''(x_\nu)=y_\nu'' \quad (\nu=1, 2, \cdots, n)$$

are given. They prove the following two theorems. (a) Let n be odd and the x_ν form a symmetric system. Then in general there is no solution; and if there is any, there are infinitely many. (b) Let n be even and x_ν be the zeros of $(1-x^2)P'_{n-1}(x)$. Then $f(x)$ is uniquely determined. —In the remaining part of the paper the "fundamental polynomials" of the interpolation in (b) are evaluated in explicit form. *G. Szegö* (Stanford, Calif.)

2139:

Malliavin, Paul. Approximation polynomiale pondérée et produits canoniques. C. R. Acad. Sci. Paris **247** (1958), 2090–2092.

Let E be a closed set on the real axis. The non-decreasing function $m(t)$ is said to be in H^+ [or H^-] if $dm(t)=0$ outside E and $\int_E \log|1-x/t|\,dm(t) \geq 0$ [or ≤ 0]. Examples of functions in H^+ and H^- are given, for instance $\exp\{\frac{1}{2}\int_1^x \phi(t)dt/t\} \in H^+$, where $\phi(t)$ is the characteristic function of E. The author states a connection between the closure problem for $\{p(x)x^n\}$ $(n=0, 1, 2, \cdots, p(x)>0$, continuous on $E)$ in $C_0(E)$, the space of continuous functions defined on E and tending to 0 as $|x| \to \infty$, and the question whether there are functions $m(t)$ in H^+ (or H^-) having certain growth properties (depending on $p(x)$). *W. H. J. Fuchs* (Ithaca, N.Y.)

2140:

★**Sansone, G. Orthogonal functions.** Revised English ed. Translated from the Italian by A. H. Diamond; with a foreword by E. Hille. Pure and Applied Mathematics, Vol. IX. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1959. xii+411 pp. $11.00.

This is a translation of the first four chapters of the third edition (1952) of Sansone's second volume of Vitali's *Moderna teoria delle funzioni di variabile reale* [Zanichelli, Bologna; MR **13**, 741; cf. also MR **7**, 434, 376; **12**, 397], dealing with general theorems, Fourier series, series of Legendre polynomials and spherical harmonics, and series of Laguerre and Hermite functions. A short section on Fourier transforms was written for this translation, and an appendix lists for reference the theorems that are required from the first volume. The foreword by Hille is a spirited defence of the "classical" theory of orthogonal

series as presented in the book. {One could wish that the author had found room to mention some of the recent contributions that testify to the continued vitality of this classical theory.} *R. P. Boas, Jr.* (Evanston, Ill.)

2141:

Natanson, G. I. Some new applications of the Bernstein-Rogosinski summation method. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 185–211. (Russian)

The author investigates the Bernstein-Rogosinski summability of (1) the Lagrange interpolation polynomials with interpolation points at the roots of the Jacobi polynomials, (2) expansions in Hermite polynomials, (3) expansions in Laguerre polynomials, (4) expansions in eigen-functions of Sturm-Liouville second order differential equations, (5) expansions in ('Bernstein'-) polynomials orthonormal on $(-1, 1)$ with respect to the weight-function $(1-x^2)^{-(1/2)}w(x)$. A typical result: Let $0<\alpha\leq w(x)\leq\beta$. Let $w(x)$ have modulus of continuity less than $K[\log(1/\delta)]^{-1-\epsilon}$. Let

$$x_1 = x\cos(\pi/2n)-(1-x^2)^{1/2}\sin(\pi/2n),$$

$$x_2 = x\cos(\pi/2n)+(1-x^2)^{1/2}\sin(\pi/2n).$$

If

$$(1-x^2)^{-(1/2)}w(x)f^2(x)\in L(-1, 1),$$

then at all Lebesgue points of $f(x)$ in $(-1, 1)$

$$\lim_{n\to\infty}\tfrac{1}{2}[S_n(x_1)+S_n(x_2)] = f(x),$$

where S_n is the nth partial sum of the Fourier expansion of $f(x)$ in 'Bernstein'-polynomials.

W. H. J. Fuchs (Ithaca, N.Y.)

2142:

Erdös, P. Concerning approximation with nodes. Colloq. Math. **6** (1958), 25–27.

Soit $f(x)$ une fonction continue dans $[-1, +1]$, E_n sa meilleure approximation par un polynome P_n de degré n, E_n' sa meilleure approximation par un polynome P_n tel que $P_n(0)=f(0)$; on a: $E_n\leq E_n'\leq 2E_n$. L'auteur prouve que, pour la fonction

$$f(x) = \sum_1^\infty T_{2n_k}(x)/k!,$$

où T_n désigne le $n$ième polynome de Tchebitcheff, on a:

$$\limsup(E_n'/E_n) = 2,$$

pourvu que les n_k croissent assez rapidement.

J. Favard (Paris)

2143:

Burov, V. N. Some effective solution methods for Čebyšev problems of best approximation. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 731–733. (Russian)

[A translation by the reviewer is available from Special Libraries Assoc. Transl. Centers, The John Crerar Library, 86 E. Randolph St., Chicago 1, Ill.]

The minimax approximation f_F to a continuous function $f(x)$, $a\leq x\leq b$, by functions of an n-parameter solvent-unisolvent family F [Motzkin, Bull. Amer. Math. Soc. **55** (1949), 789–793; MR **11**, 101], here called interpolation class [Morozov, Izv. Akad. Nauk SSSR. Ser. Mat. **16** (1952), 75–100; MR **13**, 728], is found by modifications of two iterative procedures of Remez [E. Remez, *On the methods for best approximate representation of functions in the sense of Tchebyshev*, Verlag Ukrain. Akad. Wiss., Kiev, 1935; Dokl. Akad. Nauk SSSR **101** (1955), 409–412; MR **17**, 477; Novodvorskiĭ and Pinsker, Uspehi Mat. Nauk (N.S.) **6** (1951), no. 6 (46), 174–181 (translated at NYU); MR **13**, 728], one of which applies only to linear families F, while the other converges merely to an approximation of f_F. *T. S. Motzkin* (Los Angeles, Calif.)

2144:

Sun', Yun-šen. On the best approximation of classes of functions representable in the convolution form. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 247–250. (Russian)

Let $f_k(x)$, $k=1, \cdots, n$ be ω-periodic continuous functions which form a Čebyšev system for the interval $[0, \omega]$. For functions $(*)$ $f(x)=\omega^{-1}\int_0^\omega K(t-x)\varphi(t)dt$ with $\|\varphi\|_p\leq 1$, $K\in L^q$, $p^{-1}+q^{-1}=1$, $1<p\leq+\infty$, the author defines $M_n^{(p)}=\sup_\varphi E_n(f)$, where $E_n(f)$ is the degree of the best uniform approximation of $f(x)$ by linear combinations of the f_k. He finds an explicit expression for $M_n^{(p)}$, similar to formulas given by Nikol'skiĭ [Izv. Akad. Nauk SSSR. Ser. Mat. **10** (1946), 207–256; MR **8**, 149]. He discusses the uniqueness of an extremal function f of the class $(*)$ for which $M_n^{(p)}=E_n(f)$, and the uniqueness of the corresponding φ. For the case when $E_n(f)$ is the degree of the best uniform approximation of a function $(*)$ by trigonometric polynomials T_{n-1} of order not exceeding $n-1$, he gives conditions for the equality to hold in the obvious relation

$$\sup_\varphi E_n(f)\leq\pi^{-1}\min_{T_{n-1}}\left\{\int_0^{2\pi}|K(t)-T_{n-1}(t)|^q dt\right\}^{1/q}.$$

G. G. Lorentz (Syracuse, N.Y.)

2145:

Dingle, R. B. Asymptotic expansions and converging factors. I. General theory and basic converging factors. Proc. Roy. Soc. London. Ser. A **244** (1958), 456–475.

The subject of the present paper is the investigation of the following conjectures. (a) It should normally be possible to write down the remainder in a truncated asymptotic expansion $\sum_1^n a_r/z^{r-1}+R_n$ directly from the form of the coefficients a_r alone. (b) It should thus normally be possible to compute a function exactly from the complete asymptotic expansion alone, without independent knowledge of the remainder integral in the truncated expansion or knowledge of the linear differential equation (if any!) satisfied by the function. (c) The Stokes phenomenon of discontinuities in multiplying constants at certain phases of the complex variable z should normally be contained implicitly in the formal representation $\sum_1^\infty a_r/z^{r-1}$.

A set of four examples plays a fundamental role in this investigation:

(A) $a_r=(-1)^{r-1}(r-1)!$; $|\mathrm{ph}\, z|<\pi$.

$$\sum_1^\infty\frac{(r-1)!}{(-z)^{r-1}}\to\sum_1^n\frac{(r-1)!}{(-z)^{r-1}}+\frac{n!}{(-z)^n}\Lambda_n(z),$$

$$\Lambda_s(z) = \frac{1}{s!}\int_0^\infty\frac{w^s e^{-w}dw}{1+w/z}.$$

(B) $a_r=(r-1)!$; z real and positive.

$$\sum_1^\infty \frac{(r-1)!}{z^{r-1}} \to \sum_1^s \frac{(r-1)!}{z^{r-1}} + \frac{s!}{z^s}\,\overline{\Lambda}_s(-z),$$

$$\overline{\Lambda}_s(-z) = \frac{1}{s!}\,P\int_0^\infty \frac{w^s e^{-w}dw}{1-w/z}$$

(P denoting "principal value").

(C) $a_r=(-1)^{\frac{1}{2}(r-1)}(r-1)!$ for r odd, $a_r=0$ for r even; $|\mathrm{ph}\,z|<\frac{\pi}{2}$.

$$\sum_{1,3,5,\cdots}^\infty \frac{(-1)^{\frac{1}{2}(r-1)}(r-1)!}{z^{r-1}} \to \sum_{1,3,5,\cdots}^{s\ \mathrm{odd}} \frac{(-1)^{\frac{1}{2}(r-1)}(r-1)!}{z^{r-1}} + \frac{(-1)^{\frac{1}{2}(s+1)}(s+1)!}{z^{s+1}}\,\Pi_{s+1}(z),$$

$$\Pi_s(z) = \frac{1}{s!}\int_0^\infty \frac{w^s e^{-w}dw}{1+(w/z)^2}.$$

(D) $a_r=(r-1)!$ for r odd, $a_r=0$ for r even; z real.

$$\sum_{1,3,5,\cdots}^\infty \frac{(r-1)!}{z^{r-1}} \to \sum_{1,3,5,\cdots}^{s\ \mathrm{odd}} \frac{(r-1)!}{z^{r-1}} + \frac{(s+1)!}{z^{s+1}}\,\overline{\Pi}_{s+1}(iz),$$

$$\overline{\Pi}_s(iz) = \frac{1}{s!}\,P\int_0^\infty \frac{w^s e^{-w}dw}{1-(w/z)^2}.$$

The integrals Λ, $\overline{\Lambda}$, Π and $\overline{\Pi}$ are called "basic converging factors". The Stokes phenomenon is contained in the relations existing between $\overline{\Lambda}$ and Λ, and between $\overline{\Pi}$ and Π. Recurrence relations, expansions in ascending powers of z, expansions for large $|z|$ are given for the basic converging factors and their "reduced" derivatives $f^{(m)}(z)=\frac{1}{m!}\left(\frac{d}{dz}\right)^m f(z)$ at $z=s$.

Five place tables for $\Lambda_s(z)$, $\overline{\Lambda}_s(z)$, $\Pi_s(z)$ and $\overline{\Pi}_s(iz)$ are computed for $s=-0.5(0.5)6.0$; $z=0.1(0.1)1(0.2)2(0.5)10(1)20$. *S. C. van Veen* (Delft)

2146:

Dingle, R. B. Asymptotic expansions and converging factors. II. Error, Dawson, Fresnel, exponential, sine and cosine, and similar integrals. Proc. Roy. Soc. London. Ser. A **244** (1958), 476–483.

The results of the preceding paper are extended to the case $|a_r|=\Gamma(r+\alpha)$, α and a_r real. The asymptotic expansion of the above-mentioned functions can be replaced by an easily calculable function involving one or other of the "basic converging factors" (B.C.F.) Λ, $\overline{\Lambda}$, Π or $\overline{\Pi}$, e.g.

$$\sum_1^\infty \frac{\Gamma(r+\alpha)}{(-z)^{r-1}} \to \sum_1^s \frac{\Gamma(r+\alpha)}{(-z)^{r-1}} + \frac{\Gamma(n+\alpha+1)}{(-z)^n}\,\Lambda_{s+\alpha}(z) \quad (|\mathrm{ph}\,z|<\pi).$$

Special cases: Error integral: $\mathrm{erf}\,z=\int_0^z e^{-u^2}du$; B.C.F. $=\Lambda_{s-1/2}(z^2)$; $|\mathrm{ph}\,z|<\pi/2$. Dawson's integral: $\mathrm{erfi}\,z=\int_0^z e^{u^2}du$; B.C.F. $=\overline{\Lambda}_{s-1/2}(-z^2)$. Fresnel integrals: $\mathrm{C[S]}(z)=(2/\pi)^{1/2}\int_0^{\sqrt{z}}\cos\,[\sin]\,u^2du$; B.C.F. $=\Pi_{s+1/2}(z)$; $|\mathrm{ph}\,z|<\pi/2$. Exponential integrals: $-\mathrm{Ei}\,(-z)=\int_z^\infty u^{-1}e^{-u}du$; B.C.F. $=\Lambda_s(z)$; $|\mathrm{ph}\,z|<\pi$. $\mathrm{Ei}\,(z)=P\int_{-\infty}^z \frac{e^u du}{u}$; B.C.F. $=\overline{\Lambda}_s(-z)$. Sine and cosine integrals: $\mathrm{Ci[Si]}(z)=\int_z^\infty u^{-1}\cos\,[\sin]\,udu$; B.C.F. $=\Pi_{s+1}(z)$; $|\mathrm{ph}\,z|<\pi$. Raabe integrals:

$$R_s(z) = z\int_0^\infty \frac{\sin udu}{z^2+u^2}, \quad R_c(z) = -\int_0^\infty \frac{u\cos udu}{z^2+u^2};$$

$$\text{B.C.F.} = \overline{\Pi}_{s+1}(iz).$$

Goodwin-Staton integral:

$$f(z) = \int_0^\infty \frac{e^{-u^2}du}{u+z} = e^{-z^2}\{\sqrt{\pi}\,\mathrm{erfi}\,(z)-\tfrac{1}{2}\,\mathrm{Ei}\,(z^2)\};$$

B.C.F. $=\Lambda_{s-\frac{1}{2}}(z^2)$, $\Lambda_s(-z^2)$.

By straightforward generalizations of these results, asymptotic expansions are obtained for the generalized exponential, sine and cosine integrals, incomplete gamma function, etc.; e.g., $\int_0^z u^p e^{-u}du$; B.C.F. $=\Lambda_{s-p-1}(z)$.

Utilizing the tables of I, these results yield with $n=4$, $z=4$ in most cases a value correct to seven significant figures. *S. C. van Veen* (Delft)

2147:

Dingle, R. B. Asymptotic expansions and converging factors. III. Gamma, psi and polygamma functions, and Fermi-Dirac and Bose-Einstein integrals. Proc. Roy. Soc. London. Ser. A **244** (1958), 484–490.

In this paper the asymptotic expansions $\sum_1^\infty a_r/z^{r-1}$ are those for which $|a_r|=\Gamma(r+\alpha)f(r-1)$, where $f(z)$ is a function which rapidly approaches a constant value as r is increased. Special cases are: The gamma function with B.C.F. $=$

$$\sum_{\mu=1}^\infty \frac{1}{\mu^{s+3}}\,\Pi_{s+1}(2\pi\mu z)\Big/\sum_{\mu=1}^\infty \frac{1}{\mu^{s+3}}, \quad |\mathrm{ph}\,z|<\frac{\pi}{2}.$$

The psi (or digamma) function $\psi(z)=d\,\ln\,\Gamma(z)/dz$, B.C.F. $=$

$$\sum_{\mu=1}^\infty \frac{1}{\mu^{s+3}}\,\Pi_{s+2}(2\pi\mu z)\Big/\sum_{\mu=1}^\infty \frac{1}{\mu^{s+3}}, \quad |\mathrm{ph}\,z|<\frac{\pi}{2}.$$

Fermi-Dirac integrals:

$$F_p(z) = \frac{1}{\Gamma(p+1)}\int_0^\infty \frac{u^p du}{e^{u-z}+1}.$$

Bose-Einstein integrals:

$$B_p(z) = \frac{1}{\Gamma(p+1)}\int_0^\infty \frac{u^p du}{e^{u-z}-1}$$

(for $\mathrm{Re}\,(z)>0$, z is to be treated as if real and the principal value taken). In regard to the Fermi-Dirac integrals the author remarks that, utilizing the tables of I, the value for $F_{1/2}(z)$ is in complete accord with the seven-figure value obtained from McDougall & Stoner's table [Phil. Trans. A **237** (1938), 67–104]. The simplicity of the calculation from the new formula given here, taking only a matter of minutes if the tables of I are at hand, may be contrasted with McDougall & Stoner's remark that the evaluation of an essentially equivalent function (by numerical integration) even for a small value of z (e.g. $z=2$) may require about eighteen hours.

S. C. van Veen (Delft)

2148a:

Dingle, R. B. Asymptotic expansions and converging factors. IV. Confluent hypergeometric, parabolic cylinder,

modified Bessel, and ordinary Bessel functions. Proc. Roy. Soc. London. Ser. A **249** (1959), 270–283.

2148b:

Dingle, R. B. Asymptotic expansions and converging factors. V. Lommel, Struve, modified Struve, Anger and Weber functions, and integrals of ordinary and modified Bessel functions. Proc. Roy. Soc. London. Ser. A **249** (1959), 284–292.

2148c:

Dingle, R. B. Asymptotic expansions and converging factors. VI. Application to physical prediction. Proc. Roy. Soc. London. Ser. A **249** (1959), 293–295.

The first three papers of this series [reviewed above] gave a theory in which the divergent 'remainders' of asymptotic expansions are replaced by easily calculable series involving one or other of four 'basic converging factors'. In the present papers, IV, V, this theory is applied to the functions named above. The numerical power of the formulae is illustrated; for example $J_1(2)$ and $Y_1(2)$ are found correct to 6+ decimals.

In VI a physical effect relating to diamagnetism is 'predicted' by transformation of a divergent series.

T. M. Cherry (Melbourne)

2149:

Jones, Douglas S.; and Kline, Morris. Asymptotic expansion of multiple integrals and the method of stationary phase. J. Math. Phys. **37** (1958), 1–28.

The paper is concerned with asymptotic expansions, for large k, of integrals of the type

$$J = \iint_D g(x, y)e^{ikf(x,y)}dydx, \tag{1}$$

where g and f are real functions possessing derivatives of all orders, at all points interior to and on the boundary of D. Such integrals occur frequently in diffraction theory of both visible and microwave optics.

For sufficiently large k the main contributions to J come from certain "critical points", such as points within D where f is stationary, or on the boundary of D where the tangential derivative of f vanishes. In the present paper the asymptotic expansion of J is obtained in an elegant manner by first transforming the double integral into a single Fourier integral with the help of the Dirac delta function, and then applying to it known results relating to the asymptotic behaviour of single Fourier integrals. The contributions from the critical points of the various types are obtained. The results are substantially in agreement with those obtained by J. Focke [Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Nat. Kl. **101** (1954), no. 3; MR **16**, 919] using a different method.

It is claimed that the calculation of the successive coefficients of the various asymptotic series from the several critical points is simpler by the present method than by Focke's. The possibility of applying the method to asymptotic expansions of higher dimensional multiple integrals of the form (1) is noted.

Readers interested in this problem should also consult recent publications by N. Chako [C.R. Acad. Sci. Paris **247** (1958), 436–438, 580–582, 637–639; MR **20** #1878, 5017, 5018].

E. Wolf (Manchester)

FOURIER ANALYSIS

See also 1954, 2043, 2044, 2140, 2144, 2176, 2187.

2150:

Robertson, M. S. Cesàro partial sums of harmonic series expansions. Pacific J. Math. **8** (1958), 829–846.

Let the harmonic function $v(r, \theta)$ have the sine series expansion

$$v(r, \theta) = \sum_1^\infty a_\nu r^\nu \sin \nu\theta, \tag{*}$$

convergent for $0 \leq r < 1$, and let $v(r, \theta) \geq 0$ for $0 < r < 1$, $0 < \theta < \pi$. Denote by $S_n^{(k)}(r, \theta)$ the nth Cesàro partial sum of (*). Then, for $k = 0, 1, 2, 3$, the author proves that there exists a number $R_n^{(k)}$ depending on n and k only, such that $S_n^{(k)}(r, \theta) \geq 0$ for $0 \leq r \leq R_n^{(k)}$, $0 \leq \theta \leq \pi$, but not always for $r > R_n^{(k)}$. Asymptotic expansions for $R_n^{(k)}$ are obtained, which are complicated to state in full; for instance,

$$R_n^{(1)} = 1 - 2n^{-1} \log n + n^{-1} \log \log n - n^{-1} (\log \mu + o(1)),$$

where $\mu = 1$ when n is even and, when n is odd, $\mu = \max |h^{-1} \sin h|$ for $\pi \leq h \leq 3\pi/2$. These results extend and sharpen work of Fejér [J. London Math. Soc. **8** (1933), 53–62] and Szász [Trans. Amer. Math. Soc. **52** (1942), 12–21; MR **4**, 37]. Applications are given to the question of the schlichtness of partial sums of certain schlicht functions.

P. B. Kennedy (Cork)

2151:

Ul'yanov, P. L. Unconditional convergence and summability. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 811–840. (Russian)

The following theorem generalizes in several ways the much simpler fact that each trigonometric series $\sum (a_n \cos nx + b_n \sin nx)$ for which

$$\sum |a_n \cos nx + b_n \sin nx| < M$$

must be the Fourier series of a function in the Lebesgue class L_2. Let (a_{nm}) be a matrix for which

$$\lim_{n\to\infty} \sum_{m=0}^\infty a_{nm} = 1, \quad \lim_{n\to\infty} a_{nm} = 0.$$

Let $\sum (a_n \cos nx + b_n \sin nx)$ be a trigonometric series such that to each permutation $p_0, p_1, \cdots$ of the integers $0, 1, \cdots$ there correspond a positive number A, a set E having positive measure, and an index N such that

$$\limsup_{k\to\infty} \left| \sum_{m=0}^k a_{nm} \sum_{i=0}^m (a_{p_i} \cos p_i x + b_{p_i} \sin p_i x) \right| \leq A,$$

whenever $x \in E$ and $n > N$. Then the trigonometric series is the Fourier series of a function in L_2 over $0 \leq x \leq 2\pi$. Several results related to this theorem are given. The last part of the paper treats orthogonal series $\sum c_k \phi_k(x)$ where the functions $\phi_0, \phi_1, \cdots$ form an orthonormal set over the interval $0 \leq x \leq 1$. If $\liminf |c_k| = 0$, if E is a set having positive measure, and if each rearrangement of $\sum a_k \phi_k(x)$ converges almost everywhere in E by some matrix $(a_{n,m})$ of the type described above, then each rearrangement of $\sum a_k \phi_k(x)$ converges almost everywhere in E. If the functions $\phi_k(x)$ are uniformly bounded and each rearrangement of $\sum c_k \phi_k(x)$ converges almost everywhere in $0 \leq x \leq 1$, then $\sum |c_k|^2 < \infty$. There exist a complete orthonormal set $\phi_1, \phi_2, \cdots$ and coefficients $c_1, c_2, \cdots$ such that

lim sup $|c_k| = \infty$ and $\sum c_k\phi_k(x)$ converges absolutely (and hence unconditionally) over $0 \leq x \leq 1$ to a function $F(x)$ which lies in none of the classes L_p for which $p > 0$. The paper concludes with an example of an orthogonal series $\sum c_k\phi_k(x)$ which converges to a function $F(x)$ and which possesses a rearrangement which converges to a function $F_2(x)$ for which $F_2(x) \neq F_1(x)$ over $0 \leq x \leq 1$.

R. P. Agnew (Ithaca, N.Y.)

2152:

Men'šov, D. E. On limit functions of a trigonometric series. Trudy Moskov. Mat. Obšč. **7** (1958), 291–334. (Russian)

Let $Q_n(x)$ be the partial sums of a trigonometric series; $\varphi(x)$ is a limit function of the series on a set E of positive measure if there is a subsequence of $\{Q_n(x)\}$ that converges to $\varphi(x)$ almost everywhere on E. Let M be a set of measurable functions, each defined on some set E of positive measure (possibly differing from function to function). The author finds a necessary and sufficient condition on M so that M will be the set of limit functions of some trigonometric series, namely that M is "closed in the narrow sense", where this term is defined as follows. Consider sequences $\{\varphi_n(x)\}$ defined on sets E_n, let lim inf $E_n = Q$ which differs from E only on a set of measure zero, and call φ a limit element of $\{\varphi_n\}$ in the wide sense if $\varphi_n(x) \to \varphi(x)$ almost everywhere on E. Then M is closed in the narrow sense if it contains all limit elements in the wide sense of sequences of its elements. The main theorem is deduced from two others, one of which deals with sequences of measurable functions in general while the other deals with the construction of trigonometric series with assigned limit functions. *R. P. Boas, Jr.* (Evanston, Ill.)

2153:

Loo, Ching-tsün. On a problem of Zygmund. Sci. Record (N.S.) **1** (1957), 363–368.

It was proved by Zygmund [J. London Math. Soc. **3** (1928), 194–196] that the Fourier series of a continuous function is absolutely convergent if its modulus of continuity satisfies $\omega(\delta) \leq C \log^{-\eta}(1/\delta)$, where $\eta > 2$. The author shows that this result no longer holds when $\eta = 2$. However, if $1 < \eta \leq 2$, such a series is absolutely (C, α)-summable for $\alpha > \eta^{-1} - \frac{1}{2}$.

W. W. Rogosinski (Newcastle-upon-Tyne)

2154:

Gupta, D. P. On the Cesàro summability of the ultraspherical series. I. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 269–278.

In general, the summability properties of trigonometric Fourier series are particular cases of corresponding properties of the family of ultraspherical expansions with parameter λ, since for $\lambda \to 0$ the ultraspherical series reduces to Fourier trigonometric series.

This paper generalizes to any λ in the interval $0 < \lambda < 1$ the theorems on Cesàro summability which F. T. Wang proved for the trigonometric series. The particular case $\lambda = \frac{1}{2}$ yields new results for the Laplace series. The question whether the limitation $\lambda < 1$ is related to the method of proof or is an essential one is left open.

E. Kogbetliantz (New York, N.Y.)

2155:

Sun', Yun-Šen. Best approximation of periodic differentiable functions by trigonometric polynomials. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 67–92. (Russian)

Let $W^{(r)}(\alpha)$ be the class of functions of the form $f(x) = a_0 + \pi^{-1}\int_0^{2\pi} K(t-x)\varphi(t)dt$, where $K(t) = \sum_{n=1}^{\infty} n^{-r}\cos(nt - \frac{1}{2}\pi\alpha)$, $r > 0$, α is real, and $\varphi(t)$ is an integrable function with $\int_0^{2\pi}\varphi dt = 0$, and $|\varphi(t)| \leq 1$ almost everywhere. For $r \geq 6$, the author finds that $\sup_{f \in W^{(r)}(\alpha)} E_n(f)$, the degree of the uniform approximation of f by trigonometric polynomials of order $n-1$, is equal to $4\pi^{-1}n^{-r}K_{r,\alpha}$ with

$$K_{r,\alpha} = \left| \sum_{k=0}^{\infty} (2k+1)^{-r} \sin\left[(2k+1)\beta - \tfrac{1}{2}\pi\alpha\right] \right|,$$

β being defined by $\sum_{k=0}^{\infty}(2k+1)^{-r}\cos[(2k+1)\beta - \frac{1}{2}\pi\alpha] = 0$, $0 \leq \beta < \pi$. A similar result for $W^{(r)}(r)$, $r < 1$ was obtained by Dzyadyk [same Izv. **17** (1953), 135–162; MR **14**, 867]. If $f \in W^{(r)}(r)$, $r \geq 6$ and a continuous derivative $f^{(r)}$ exists, then $E_n(f) \leq 4\pi^{-1}n^{-r}K_{r,r}E_n(f^{(r)})$.

G. G. Lorentz (Syracuse, N.Y.)

2156:

Peyerimhoff, Alexander. Über trigonometrische Reihen mit monoton fallenden Koeffizienten. Arch. Math. **9** (1958), 75–81.

R. P. Boas [voir A. Zygmund, Trigonometrical Series, 2nd ed., Cambridge Univ. Press, New York, 1959; Vol. I, p. 229] a montré que $x^{-\gamma}f(x) \in L(0, \pi)$ si et seulement si $\sum n^{\gamma-1}\lambda_n$ converge, où $0 < \gamma \leq 1$, $f(x) = \sum_1^{\infty}\lambda_n \sin nx$, $\lambda_n \downarrow 0$. Ce résultat a été généralisé dans de diverses directions. L'auteur donne ici une généralisation qui contient toutes les autres. Soit $\eta(x) \geq 0$ $(0 \leq x \leq \pi)$ avec $x\eta(x) \in L(0, \pi)$; alors de

$$(*) \qquad \sum_1^{\infty} n\lambda_n \int_0^{1/n} x\eta(x)dx < \infty,$$

résulte (**) $\eta(x)f(x) \in L(0, \pi)$. La réciproque est vrai sous une condition supplémentaire imposée à $\eta(x)$. Toutes ces conditions sont remplies, si par example $\eta(x) = x^{-\gamma}$, ou bien $\eta(x) = x^{-\gamma}f(1/x)$ $(0 \leq \gamma < 2)$, où $f(t)$ désigne une fonction à croissance lente [voir A. Zygmund, ibid., pp. 186–191]. Pourtant, comme l'auteur a montré, il existe des fonctions $\eta(x) \in L(0, \pi)$ et des suites $\lambda_n \downarrow 0$ telles que (**) a lieu, tandis que (*) n'est pas remplie. *M. Tomić* (Belgrade)

2157:

Shapiro, H. S. The expansion of mean-periodic functions in series of exponentials. Comm. Pure Appl. Math. **11** (1958), 1–21.

L'auteur étudie les fonctions moyenne-périodiques analytiques d'une variable complexe. Une telle fonction $f(z)$ est solution d'une équation de convolution

$$\int f(z+\alpha)dT(\alpha) = 0,$$

où T est une distribution à support compact. La "fonction caractéristique" est $\varphi(w) = \int e^{\alpha w}dT(\alpha)$; l'auteur introduit aussi la fonction

$$\Phi_\lambda(w) = \int dT(\alpha) \int_{\lambda-\alpha}^{\lambda} f(z+\alpha)e^{-wz}dz.$$

A toute fonction f on peut associer un développement

formel en série d'exponentielles-polynomes: $f(z)=\sum_{\nu}' P_\nu(z)e^{w_\nu z}$, où les w_ν sont les zéros de φ. Les termes de la série sont alors donnés par la formule

$$P_\nu(z)e^{w_\nu z} = \frac{1}{2i\pi} \text{ Rés. } \frac{\Phi_\lambda(w)}{\varphi(w)} e^{wz}.$$

En utilisant la majoration exponentielle de φ et Φ, et le théorème du minimum du module sur une suite de circonférences, l'auteur démontre que la série formelle de f converge absolument et uniformément par groupements de termes sur tout compact du plan complexe. En se bornant aux équations différentielles aux différences finies, du type $\sum_i Q_i(d/dz)f(z+\alpha_i)=0$, il donne des conditions suffisantes pour que la série formelle de f converge aussi sans groupements de termes, et il étudie l'algèbre des opérateurs différentiels aux différences finies à partir de résultats de Ritt. *L. Schwartz* (Paris)

2158:

Wiener, N.; and Akutowicz, E. J. A factorization of positive Hermitian matrices. J. Math. Mech. **8** (1959), 111–120.

Let $H(\theta)=(h_{jk}(\theta))$ be a Hermitian positive-definite matrix function defined on $[0, 2\pi]$ such that each $h_{jk}\in L_1(0, 2\pi)$ for all j and k. Suppose further that $-\infty < \int_0^{2\pi} \log \det H(\theta)d\theta$. Then $H(\theta)$ can be factored as $H(\theta)=N(\theta)N(\theta)^*$, where $N(\theta)=(n_{jk}(\theta))$ is such that $n_{jk}\in L_2(0, 2\pi)$ for all j, k and the Fourier coefficients of n_{jk} are zero for all negative values. This generalizes a theorem of G. Szegő [Math. Z. **6** (1920), 167–202], which is precisely the one-dimensional case. The theorem proved here has also been obtained by Wiener and Masani [Acta Math. **98** (1957), 111–150; MR **20** #4323; p. 148]. A more general theorem with somewhat weaker hypotheses and a stronger conclusion has been proved by Helson and Lowdenslager [Acta Math. **99** (1958), 165–202; MR **20** #4155]. *E. Hewitt* (Seattle, Wash.)

2159:

★Hewitt, Edwin. A survey of abstract harmonic analysis. Some aspects of analysis and probability, pp. 105–168. Surveys in Applied Mathematics. Vol. 4. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xi+243 pp. $9.00.

This survey article is mostly about locally compact Abelian groups G. Its central theme is the structure of two convolution algebras, $L_1(G)$ and the algebra of measures on G, together with the multiplicative algebras of their Fourier transforms. Unfortunately, this survey appeared just before Malliavin's disproof of spectral synthesis [C. R. Acad. Sci. Paris **248** (1959), 2155–2157]. Malliavin has shown that if G is not compact, then $L_1(G)$ always has a closed ideal that is not the intersection of regular maximal ideals. In addition to its central theme, the survey contains some remarks about the non-abelian case, about commutative semi-simple Banach algebras, about generalizations of positive definite functions, and about $L_p(G)$ for $1<p\leq 2$. There is a large bibliography.

W. F. Stinespring (Princeton, N.J.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 2171.

2160:

Steinberg, Jacob. Transformations intégrales qui laissent invariant tout polynôme. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 69–76.

A set of functions $\{I(x)\}$ is determined for which

$$(1) \qquad \int_{-\infty}^{+\infty} I(s-t)p(t)dt = p(s)$$

for all polynomials $p(s)$; or equivalently, for which the moment problem

$$\int_{-\infty}^{+\infty} I(x)x^n dx = \begin{cases} 1, & n=0 \\ 0, & n=1, 2, \cdots \end{cases}$$

is solved. A formal argument is given, and then validated by imposing certain conditions on the integral transform

$$Kf = \int_{-\infty}^{+\infty} K(bs-t)f(t)dt,$$

namely: $K(x)$ satisfies $|K(x)|<Ce^{-q|x|}$ $(C>0, q>0)$ for all real x, and $\int_{-\infty}^{\infty} K(x)dx=1$. [See #2167 below for analogous conditions on a complex-valued K.] Define $i_m=b^m\{(b^m-1)\cdots(b-1)\}^{-1}$ (for the case $b>1$), and set $I(s, t)=\sum_{m=1}^{\infty} i_m K_m(b^m s-t)$, where

$$K_m(x) = \int_{-\infty}^{+\infty} K_{m-1}(u)K(x-bu)du.$$

The series converges absolutely and uniformly for real t, s, and $I(x)$ defined by $I(x)=I(0, -x)$ has the desired property (1). *I. M. Sheffer* (University Park, Pa.)

2161:

Zarhina, R. B. On the two-dimensional problem of moments. Dokl. Akad. Nauk SSSR **124** (1959), 743–746. (Russian)

Let $\{\mu_{m,n}; 0\leq m, n<\infty\}$ be a sequence of real numbers with the property that

$$(1) \qquad \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i\alpha_j\mu_{m_i+m_j, n_i+n_j} \geq 0$$

for every finite set $\{\alpha_i\}$ of real numbers and points $\{(m_i, n_i)\}$. The principal new result of this report is that $\{\mu_{m,n}\}$ is not necessarily a moment sequence; i.e., there may exist no bounded non-negative measure $d\sigma$ such that (2) $\mu_{m,n}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n d\sigma(x, y)$.

Let P be a polynomial in x and y; i.e., $P(x, y)=\sum \alpha_{m,n}x^m y^n$. Set $f(P)=\sum \alpha_{m,n}\mu_{m,n}$; then $f(P)$ is a positive linear functional on the linear manifold R of real polynomials of two variables in the sense that $f(P^2)\geq 0$. Conversely, if f is a positive linear functional on R, then $\mu_{m,n}=f(x^m y^n)$ satisfies (1). If $\{\mu_{m,n}\}$ is a moment sequence, then f must satisfy the additional condition that if $P(x, y)\geq 0$, then $f(P)\geq 0$.

To prove the main result the author uses the well-known theorem of Hilbert that not every non-negative polynomial in two variables may be written as a sum of squares of polynomials. Let R_n $(n>6)$ be the manifold of polynomials of degree n, Π_n the non-negative polynomials and K_n the cone consisting of sums of squares. In the

author's words: "Introduce in R_n the euclidean norm." The crucial point, for which no proof is indicated, is that the closure of K_n is not all of Π_n. Using this fact one may easily construct a linear functional which is positive on K_n, but negative on certain elements of Π_n. This answers, in the negative, a question posed by the reviewer [Duke Math. J. **24** (1957), 481–498; MR **19**, 1047].

The author also gives a sufficient condition under which a sequence satisfying (1) is a moment sequence; a result which is already contained in a theorem due to the reviewer [ibid.]. *A. Devinatz* (St. Louis, Mo.)

2162:

Buschman, R. G. An integral transformation relation. Proc. Amer. Math. Soc. **9** (1958), 956–958.

Suppose that we have two integral transformations

$$\int_0^\infty K(s,t)k(t)F[g(t)]dt = \mathscr{K}\{k(t)F[g(t)]\}$$

and

$$\int_0^\infty S(u,p)F(p)dp = \mathscr{S}\{F(p)\} = f(u).$$

The author shows how to find a $\Phi(s,u)$ so that

$$\text{(i)} \qquad \mathscr{K}\{k(t)F[g(t)]\} = \int_0^\infty \Phi(s,u)f(u)du.$$

This paper generalises Theorem 2 of an earlier paper [Pacific J. Math. **7** (1957), 1529–1533; MR **19**, 1051]. Formula (i) includes as a special case the formula of Rooney [Proc. Amer. Math. Soc. **8** (1957), 883–886; MR **19**, 854]. Two formulae connected with the parabolic cylinder functions are also obtained.

J. L. Griffith (Kensington)

2163:

Kasuga, Takashi. An example of kernel of non-Carleman type. Proc. Japan Acad. **33** (1957), 521–524.

The author proves the existence of a symmetric positive kernel $S(x,y)$ defined for $0\leq x\leq 1$, $0\leq y\leq 1$, of Baire class 1, which yields a bounded transformation in $L^{(2)}$, $Tf=\int_0^1 S(x,y)f(y)dy$, but for which $S(x,y)$ is not in $L^{(2)}$ $[0\leq y\leq 1]$ for $0\leq x\leq 1$. (T has domain all of $L^{(2)}$.)

F. J. Murray (New York, N.Y.)

2164:

Nečas, Jindřich. Une note sur la propriété caractéristique de la transformée de Laplace d'une fonction et sur certains espaces de Hilbert $\bar{H}_M$ dont la somme $\sum_{k=0,l=0}^{\infty}\bar{H}_M$ est l'ensemble des transformées de Laplace de distributions. Časopis Pěst. Mat. **83** (1958), 160–170. (Czech. Russian and French summaries)

In this paper the author defines a certain class of generalized functions ("distributions"), which equal zero at $t<0$, and for which it is possible to define a special kind of Laplace transform. The transforms of these generalized functions have to be the analytic functions $F(p)$ in the domain where $\mathrm{Re}\,p>l$ $(l=0,1,2,\cdots)$ and for which exists a whole number $k\geq 0$ such that $F(p)/p^{k+1}$ is also the Laplace transform of a function. The author proves a theorem on the necessary and sufficient conditions for the above described regular function $F(p)$ to be the Laplace transform of a function. He shows also that the set of transforms $F(p)$ could be represented by a sum of certain Hilbert spaces. The author claims that his Hilbert spaces could be very useful in proving the existence and the uniqueness of solution in problems involving partial differential equations, which he demonstrates on a simple boundary value problem involving the one-dimensional heat equation. *T. Leser* (Aberdeen, Md.)

2165:

Mikusiński, J. Sur la fonction dont la transformée de Laplace est e^{-s^α}. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 691–693.

Let

$$F_\alpha(t) = \frac{1}{2\pi i}\int_{-t\infty}^{t\infty} e^{-tz-z^\alpha}dz \quad (0<\alpha<1).$$

The author bases his study of $F(t)$ on the representation

$$F_\alpha(t) = \frac{1}{\pi}\frac{\alpha}{1-\alpha}\frac{1}{t}\int_0^\pi ue^{-u}d\varphi,$$

with

$$u = t^{-\alpha/(1-\alpha)}\left(\frac{\sin\alpha\varphi}{\sin\varphi}\right)^{1/(1-\alpha)}\frac{\sin(1-\alpha)\varphi}{\sin\varphi}.$$

This formula shows that $F_\alpha(t)>0$ when $t>0$. It is also suitable for a discussion of the asymptotic character of $F_\alpha(t)$ when $t\to 0$ or ∞. Thus

$$F_\alpha(t) \sim Kt^{-(2-\alpha)/(2-2\alpha)}\exp\left(-At^{-\alpha/(1-\alpha)}\right) \quad (t\to 0),$$

$$F_\alpha(t) \sim Mt^{-1-\alpha} \quad (t\to\infty),$$

where $A=(1-\alpha)\alpha^{\alpha/(1-\alpha)}$, $K=[2\pi(1-\alpha)]^{-1/2}\alpha^{1/(2-2\alpha)}$, $M=\pi^{-1}\sin\alpha\pi\Gamma(1+\alpha)$. *E. Hille* (New Haven, Conn.)

2166:

Saksena, K. M. Inversion and representation theorems for a generalized Laplace integral. Pacific J. Math. **8** (1958), 597–607.

In a former paper [Nieuw Arch. Wisk. (3) **6** (1958), 1–9; MR **20** #5404] the author had dealt with the transformation

$$F(x) = \int_0^\infty e^{-xt/2}(xt)^{-k-1/2}W_{k+1/2,m}(xt)\phi(t)dt,$$

where $W_{k,m}$ is the Whittaker function, using a method which was based on fractional integration and was due to Erdélyi. Here he treats the Varma integral

$$F(x) = \int_0^\infty (xt)^{m-1/2}e^{-xt/2}W_{k,m}(xt)\phi(t)dt$$

in a similar way and obtains similar, but more detailed results. In theorems 1 and 2 he reduces it to the Laplace integral, giving several formulas. In theorems 3 and 4 he deduces a real inversion formula, analogous to that proved by Widder for the Laplace integral, and necessary and sufficient conditions for a given $F(x)$ to be representable by Varma's integral. Finally, in theorems 5a, b, he deduces two formulas analogous to results due to Widder.

H. Kober (Birmingham)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 2160, 2365, 2448.

2167:

Steinberg, Jacob. Application de la théorie des suites d'Appell à une classe d'équations intégrales. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 55–68.

The integral equation

$$(1) \qquad f(s)-\lambda\int_{-\infty}^{+\infty} K(bs-t)f(t)dt = 0$$

(λ, b parameters) is studied under the conditions: (a) $K(z)$ is holomorphic in a strip B of the z-plane ($z=x+iy$) parallel to and containing the real axis; (b) in B, $|K(z)| < Ce^{-q|z|}$ ($C>0$, $q>0$); (c) $\int_{-\infty}^{\infty} K(x)dx=1$; and where $b^n\neq 1$ ($n=1, 2, \cdots$), in which case the characteristic numbers are $\lambda_n=b^{-n}$ ($n=0, 1, \cdots$), and to each λ_n there is a unique polynomial $p_n(s)$ that is a characteristic function, and on norming p_n suitably, $\{p_n\}$ is a sequence of Appell polynomials. If $|b|>1$ the generating function $A(t)$ for $\{p_n(t)\}$ (defined by $A(u)e^{ut}=\sum_0^\infty p_n(t)u^n$) is holomorphic in $|u|<q|b|$ and is given by $A(u)=\prod_{n=1}^{\infty} L(b^{-n}u)$ where $L(u)=\int_{-\infty}^{\infty} e^{-ut}K(t)dt$. Correspondingly for $|b|<1$. If $|b|\neq 1$ and $L(u)\neq 0$ in $|u|<q$, there are no characteristic functions of exponential type $<q$ other than $\{p_n\}$. Some examples are given. *I. M. Sheffer* (University Park, Pa.)

2168:

Shinbrot, Marvin. A generalization of Latta's method for the solution of integral equations. Quart. Appl. Math. **16** (1958), 415–421.

Let j satisfy an ordinary linear differential equation. Then

$$(1) \qquad \phi(x) = \int_a^b j(x-t)f(t)dt$$

can be solved by a method ascribed to Latta (but in essence quite old). The author shows that the kernel $p(x)j(x)$, where $p(x)$ is a polynomial, can be formally reduced to the latter case. The example $j(x)=\log|x|$ is worked out and remarks are made about $j=|x|^{-\alpha}$, $0<\alpha<1$, when $\phi(x)$ is an exponential polynomial. *D. G. Bourgin* (Urbana, Ill.)

2169:

Koppelman, Walter. On the spectral theory of singular integral operators. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 1252–1254.

This is a note announcing without proofs results on the spectral theory of L in $L^2(a, b)$, where

$$L(x(\lambda)) = f(\lambda)x(\lambda)+\frac{1}{\pi i}\int_a^b \frac{\sqrt{(k(\lambda)k(\mu))}}{\mu-\lambda}x(\mu)d\mu,$$

f, k real, continuously differentiable in (a, b), $f'(\lambda)\pm k'(\lambda)$ with isolated zeros only, $k>0$ almost everywhere. Then the limit points of $\sigma(L)$ form the closed interval

$$[\min\{f(\lambda)-k(\lambda)\},\quad \max\{f(\lambda)+k(\lambda)\}].$$

Isometric transformations are given, which transform L into the canonical form. Under certain conditions L will have a simple continuous spectrum and the isometries furnish a complete spectral representation for L.

František Wolf (Berkeley, Calif.)

2170:

Padmavally, Komarath. On a non-linear integral equation. J. Math. Mech. **7** (1958), 533–555.

The paper is devoted to a generalization of two papers: one by W. R. Mann and Wolf [Quart. Appl. Math. **9** (1951), 163–184; MR **13**, 134] and the other by J. H. Roberts and Mann [Pacific. J. Math. **1** (1951), 431–445; MR **13**, 354]. It studies a conduction of heat problem in a semi-infinite bar $x\geq 0$ with a non-linear boundary condition

$$U_{xx}=U_t,\quad U(x, 0)=0,\quad -U_x(0, t)=g[U(0, t), \phi(t)].$$

$\phi(t)$ is the prescribed temperature of the heated end, g governs the rate of flow. The conditions on these functions are: g continuous in both variables, strictly decreasing in the first and strictly increasing in the second and $g(u, u)=0$. The discontinuities of ϕ are discrete and ϕ bounded. In the two papers referred to it is assumed that $\phi\equiv 1$. Similarly as before the differential problem is shown to be equivalent to the singular integral equation

$$y(t) = \int_0^t \frac{G(y(\tau), t)}{\sqrt{\pi(t-\tau)}}d\tau;$$

where $G(u, t)$ is Lebesgue integrable in t in any finite interval, non-increasing with respect to u, bounded and uniformly continuous in any finite rectangle $u_1\leq u\leq u_2$, $0\leq t\leq t_0$; and for each $t>0$ there exists a finite value $u(t)$ such that $G(u(t), t)=0$, $G(u, t)>0$ for $u<u(t)$, and $u(t)$ bounded on any finite interval of t. Then it is shown that the integral equation admits a unique solution $y(t)$ such that as a functional $y(G)$, it is non-decreasing. If $u(t)$ for $0\leq t\leq T$ is within (m, M), then $y(t)$ is within the same limits. If G is non-decreasing with respect to t and $G(0, 0)=\lim_{t\to 0} G(u, t)|_{u=0}>0$, then $y(t)$ is non-decreasing with respect to t. If $\phi(0)>0$ and ϕ non-decreasing, then $U(0, t)$ is non-decreasing, $0\leq U(0, t)\leq\phi(t)$, and $U(0, t)<\phi(t)$ whenever $\phi(t)>0$. *František Wolf* (Berkeley, Calif.)

2171:

Dinghas, Alexander. Zur Existenz von Fixpunkten bei Abbildungen vom Abel-Liouvilleschen Typus. Math. Z. **70** (1958), 174–189.

Let $f(x, y)$ be a continuous function in the rectangle $0\leq x\leq a$, $|y|\leq b$, $(0<a,\ b<+\infty)$, C the set of all continuous functions $g(x)$ defined in the interval $0\leq x\leq a$ such that $|g(x)|\leq b$ and $g(0)=0$. The object of this paper is to study the fix points of the operator

$$T_\mu(g) = \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}f(t, g(t))dt,$$

where $0<\mu\leq 1$ (i.e., to find functions $g\in C$ such that $T_\mu(g)=g$ for all $0\leq x\leq a_1\leq a$). As pointed out, this problem is equivalent to the problem of solving the integro-differential equation $\{1/\Gamma(1-\mu)\cdot\int_0^x (x-t)^{-\mu}y(t)dt\}'=f(x, y)$, $y(0)=0$. For $\mu=1$ this becomes the ordinary differential equation $y'=f(x, y)$, $y(0)=0$. The principal results of this paper are: (I) If f satisfies the Lipschitz conditions $x^\mu|f(x, y_1)-f(x, y_2)|\leq\Gamma(\mu+1)|y_1-y_2|$, then the mapping T_μ has precisely one fix point which can be obtained by iteration. (For $\mu=1$ this is the classical Rosenblatt-Nagumo-Perron-van Kampen theorem for ordinary differential equations.) An example is given to show that the constant $\Gamma(\mu+1)$ is best possible. (II) If f satisfies the Lipschitz conditions $|f(x, y_1)-f(x, y_2)|\leq A\Gamma(\mu+1)|y_1-y_2|^\alpha$, $x^\mu|f(x, y_1)-f(x, y_2)|\leq k|y_1-y_2|$, where

$k\beta < 1$, $\beta = \Gamma(\mu\alpha/(1-\alpha)+1)/\Gamma(\mu/(1-\alpha)+1)$, then the mapping T_μ has precisely one fix point which can be obtained by iteration. [For $\mu = 1$ this reduces to theorems recently obtained by M. A. Krasnosel'skiĭ and S. G. Kreĭn, Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1 (67), 209–213; MR **18**, 38; and W. A. J. Luxemburg, see #2083 above.] Examples are given to show that if $k\beta \geq 1$ the theorem is no longer true. The proofs of these theorems are based on an elegant application of the Weierstrass approximation theorem.

W. A. J. Luxemburg (Pasadena, Calif.)

FUNCTIONAL ANALYSIS

See also 2031, 2091, 2094, 2108, 2109, 2110, 2134, 2480, 2481, 2482, 2483.

2172:

★**Вулих, Б. З. Введение в функциональный анализ.** [**Vulih, B. Z. Introduction to functional analysis.**] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 352 pp. 8.25 rubles.

This book, intended for technical schools, gives an elementary but adequate treatment of functional analysis for the purpose of showing how it may be applied to integral equations, best approximation, boundary value problems, approximate solution of algebraic and differential equations, generalized summation of series and so forth. There are no discussions of what might be called inner functional-analytic theorems and additional hypotheses are unhesitatingly admitted if they simplify the exposition and are likely to be fulfilled in problems of the subjects listed above. For example, there is no need for any real variable theory, since even Lebesgue integration is avoided (see below). There is very little physics in the book, and no exercises.

There are 13 chapters, whose titles, bracketed below, and approximate contents are as follows: (1) (n-dimensional Euclidean space): the author describes this chapter and the next as "propaedeutic"; (2) (infinite-dimensional Euclidean space): an elementary introduction to l^2-space; (3) (metric spaces): deals chiefly with compact sets; (4) (continuous operators in metric spaces): uses the Banach theorem on contraction operators to prove the existence of solutions of simple differential and integral equations; (5) (normed spaces): proves the existence of a polynomial of best approximation to a function continuous on a closed interval; (6) (Hilbert space): introduces the expansion of a function in the elements of a complete orthonormal set; (7) (L^2-space): gives proofs of completeness, etc., the concept of Lebesgue integration being avoided by the use of interval functions; (8) (linear operators): gives their fundamental properties and defines their Banach algebra; (9) (linear functionals): the standard representation theorems are given; weak convergence is used to demonstrate convergence of methods of mechanical quadrature; linear functionals in l^2 are used for various methods of summation of series; (10) (adjoint and self-adjoint operators in Hilbert space): various types of spectra are illustrated by integral and differential operators; (11) (completely continuous operators): discusses Fourier expansions, the Hilbert-Schmidt theory for integral equations with symmetric kernel, and Sturm-Liouville problems; (12) (approximate solution of functional equations): discusses the Ritz method, and the Bubnov-Galerkin modification; (13) (partially ordered normed spaces): the general theory [cf. e.g. the book by Kantorovič, Vulik and Pinsker reviewed in MR **12**, 340] is followed by an application to integral equations.

The book is written in very lucid style throughout.

S. H. Gould (Providence, R.I.)

2173:

Kelley, J. L. Hypercomplete linear topological spaces. Michigan Math. J. **5** (1958), 235–246.

Let F be a locally convex Hausdorff linear topological space. The author calls F hypercomplete if its convex circled subsets form a complete space under the Hausdorff uniform structure, and he characterises this property in terms of filter bases in F consisting of convex circled sets. He proves that F is hypercomplete if and only if every convex circled subset A of the adjoint (dual) space F^*, whose intersection with each equicontinuous set is weak* closed in it, is itself weak* closed. If A is only required to be a vector subspace of F^*, this property becomes the one used by V. Pták to define a fully complete space [Czechoslovak Math. J. **3** (78) (1953), 301–364; MR **16**, 262]. The author characterises fully complete spaces F also in terms of filter bases of convex circled subsets of F, and proves the equivalence of full completeness with various closed graph and open mapping properties, of which the following is typical: F is fully complete if and only if each linear mapping into a quotient space of F is continuous whenever it has a closed graph and its inverse maps neighborhoods of the origin into somewhere dense sets. The paper ends with some conclusions about the finest topology on F^* which agrees with the weak* topology on equicontinuous sets.

A. P. Robertson (Lawrence, Kans.)

2174:

Leader, Solomon. Separation and approximation in topological vector lattices. Canad. J. Math. **11** (1959), 286–296.

Let $\mathfrak{L}$ be a topological vector lattice, u a weak unit in $\mathfrak{L}$ and $\mathfrak{R}$ a Boolean ring of projection operators on $\mathfrak{L}$. The author gives a condition on $\mathfrak{R}$ by which the set of all Eu for E in $\mathfrak{R}$ generates $\mathfrak{L}$. A similar result in the case of topological lattice algebras and some applications in the theory of integral are given. *I. G. Amemiya* (Tokyo)

2175:

Phelps, R. R. Correction to "Subreflexive normed linear spaces". Arch. Math. **9** (1958), 439–440.

In this note a mistake in the second paragraph of the proof of Theorem 1.2 in the author's paper "Subreflexive normed linear spaces" [Arch. Math. **8** (1957), 444–450; MR **20** #6027] is corrected.

W. A. J. Luxemburg (Pasadena, Calif.)

2176:

Edwards, D. A. A class of singular integrals. Proc. London Math. Soc. (3) **9** (1959), 161–176.

Let x denote a mapping of finite or infinite real linear interval I into a Banach space X over the real or complex

field. For each compact interval $[a, b] \subseteq I$, let $E(a, b; x)$ be defined as the set of all finite sums of the form

$$\sum_{j=1}^{p} x(t_{2j}) - x(t_{2j-1}),$$

in which $a \leq t_1 < t_2 < \cdots < t_{2p} \leq b$. Let $E(I; x)$ denote the union of $E(a, b; x)$ for all $[a, b] \subseteq I$. The class of mappings x for which $E(I; x)$ is bounded is denoted by $\mathscr{D}(I)$, and that for which $E(I; x)$ is conditionally weakly compact by $\mathscr{S}(I)$. The space of scalar-valued functions of bounded variation on I, with the usual norm, is denoted by $\mathrm{BV}(I)$. The author shows first that if V denotes the mapping from X^* into the space of scalar-valued functions on I whose value at $x^* \in X^*$ is $x^*x(\cdot)$, then $x \in \mathscr{D}(I)$ if and only if $VX^* \subseteq \mathrm{BV}(I)$ and $x \in \mathscr{S}(I)$ if and only if V is a weakly compact operator from X^* into $\mathrm{BV}(I)$.

The singular integral theorem is then as follows. Let $I = [c, d]$ be a closed interval of R and $\{\varphi_n\}$ a sequence of scalar-valued functions defined on I^2 with $\varphi_n(t, \cdot)$ Lebesgue integrable over I for $n = 1, 2, 3, \cdots$, $t \in I$. Let $x \in \mathscr{S}(I)$, let $J_n(x; t)$ denote the Bochner integral $\int_c^d \varphi_n(t, u)x(u)du$, and let $\Phi(t, v) \equiv \mathrm{sign}\ (v-t) \int_t^v \varphi_n(t, u)du$ for $v, t \in I$. The following three conditions on the sequence $\{\varphi_n\}$ are then shown to be equivalent: (i) $J_n(\xi, t) \to \frac{1}{2}(\xi(t-0) + \xi(t+0))$ $(c < t < d)$ as $n \to \infty$ for each $\xi \in \mathrm{BV}(I)$; (ii) $J_n(x, t) \to \frac{1}{2}(x(t-0) + x(t+0))$ $(c < t < d)$ strongly as $n \to \infty$ for each $x \in \mathscr{S}(I)$; (iii) for each fixed interior point t of I, $\Phi_n(t, v) \to \frac{1}{2}$ boundedly for $t < v \leq d$ and also boundedly for $c \leq v < t$ as $n \to \infty$. When these conditions hold the convergence in (i) takes place uniformly (for each fixed $t \in (c, d)$) over every conditionally weakly compact subset K of $\mathrm{BV}(I)$.

As an application, the real-valued functions ξ on I^2 of bounded Fréchet variation and bounded variation in each variable separately are characterized as those for which $x \in \mathscr{D}(I)$ with $x(t) = \xi(t, \cdot)$, $t \in I$. Use is made of this example to obtain for such functions of bounded Fréchet variation the continuity properties established by M. Morse and the reviewer [Canad. J. Math. **2** (1950), 344–374; MR **12**, 247], and to generalize, using the singular integral theorem, our analogue of the Jordan test for double Fourier series [Rivista Mat. Univ. Parma **1** (1950), 3–18; MR **11**, 512]. *W. R. Transue* (Gambier, Ohio)

2177:

Isbell, J. R. Algebras of uniformly continuous functions. Ann. of Math. (2) **68** (1958), 96–125.

Let μX denote a uniform space, and let $C(\mu X)$ denote the set of all uniformly continuous real-valued functions on μX (where the real line R carries its usual metric uniformity). Then $C(\mu X)$, with the usual pointwise operations, is a vector-lattice, but need not be closed under multiplication. The author is concerned, for the most part, with those uniform spaces such that $C(\mu X)$ is closed under multiplication.

Let A denote an algebra of real-valued functions on a set X that contains the constant functions, and such that if $x_1, x_2 \in X$, and $x_1 \neq x_2$, then there is an $f \in A$ such that $0 \neq f(x_1) \neq f(x_2)$. The coarsest uniformity on X making every $f \in A$ uniformly continuous will be denoted by μ_A. If $1/f \in A$ whenever f fails to vanish on X [resp. $f \geq 1$], then A is said to be closed under inversion [resp. bounded inversion]. If for any $f_1, \cdots, f_n \in A$, and continuous real-valued function g on E^n, $g(f_1, \cdots, f_n) \in A$, then A is said to be closed under composition. Closure under composition is an algebraic invariant. If A is closed under composition, then every uniformly continuous function on $\mu_A X$ is a uniform limit of functions in A, but the uniform closure of A need not be closed under multiplication. Thus an analogue of the Stone-Weierstrass theorem holds in this context that does not hold for the ring $C(Y)$ of all continuous functions on a completely regular space Y unless Y is compact.

Moreover, the following analogue of the Tietze-Urysohn extension theorem holds. If μX is a uniform space such that (a) $C(\mu X)$ is an algebra closed under composition, and (b) μ is the weak uniformity induced by μX, then any subspace νY of μX satisfies (a) and (b), and every function in $C(\nu Y)$ has an extension in $C(\mu X)$.

If μ_A induces a locally compact and σ-compact topology on X, and A is closed under uniform convergence and inversion, then $A = C(\mu_A X)$. It follows that any function algebra A closed under uniform convergence and inversion is closed under composition.

Let $H(A)$ denote the space of isotone homomorphisms of A onto R with the weak topology, and let A^* denote the subalgebra of bounded elements of A. If A is closed under uniform convergence and inversion, then A consists of all those functions f on X such that f converges to a definite limit, finite or infinite, at every point of $H(A^*)$. (X is regarded as a subspace of $H(A^*)$; this is possible since bounded inversion implies that A^* separates points on X.)

These, and other results, are exploited to obtain a structure theory for algebras $C(\mu X)$ similar to the known structure of $C(X)$ [cf. E. Hewitt, Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126]. In particular, if M is a maximal ideal of a function algebra A closed under composition, and if M is a maximal ideal of A, then A/M is a real-closed field. {It is also stated that if, in addition, A is closed under uniform convergence, then $A/M = R$ or A/M is an η_1-set. But the author and the reviewer have discovered that this is false, and that if A is closed under composition with continuous real-valued functions on E^∞, then the conclusion is valid. For definition of η_1-set, see Erdös, Gillman, and Henriksen, Ann. of Math. (2) **61** (1955), 542–554 [MR **16**, 993].}

In a long concluding section, the author establishes a bicategorical duality between complete uniform spaces μX and the algebras $C(\mu X)$ in which subspaces correspond to quotient algebras and quotient spaces to subalgebras. The notion of bicategory here is less restrictive than in MacLane's development [Bull. Amer. Math. Soc. **56** (1950), 485–516; MR **14**, 133], so a self-contained exposition is given.

The paper abounds with instructive examples that show that the hypotheses of the main theorems cannot easily be weakened. *M. Henriksen* (Lafayette, Ind.)

2178:

Pursell, Lyle E. A note on isomorphisms of $C(X, R)$ and $C^*(X, R)$. Bull. Calcutta Math. Soc. **49** (1957), 47–48.

Let $C(X, R)$ denote the ring of all continuous real-valued functions on a completely regular space X. An alternate proof is presented of the known fact that if $C(X, R)$ and $C(X', R)$ are isomorphic, and X and X' satisfy the first axiom of countability, then X and X' are homeomorphic. *M. Henriksen* (Lafayette, Ind.)

2179:

Kelley, J. L. Averaging operators on $C_\infty(X)$. Illinois J. Math. **2** (1958), 214–223.

A linear operator T defined on the Banach algebra $C_\infty(X)$ of real valued continuous functions, vanishing at ∞, on a locally compact Hausdorff space X is called averaging if X may be decomposed into slices and on each slice $T(f)$ is an average of the value of f on this slice. It is proved that T is averaging if and only if $T(fT(g))=(Tf)(Tg)$, and if T is positive and idempotent then T is averaging if and only if the range of T is a subalgebra of $C_\infty(X)$, etc. The paper is an extension of some results of G. Birkhoff's paper, in which the underlying space X is compact [*Algèbre et théorie des nombres*, Colloq. Internat. Centre Nat. Recherche Sci. no. 24, pp. 143–153, Centre Nat. Recherche Sci., Paris, 1950; MR **13**, 361]. *Shu-Teh chen Moy* (Detroit, Mich.)

2180:

König, Heinz. Über das Wachstumsverhalten von linearen Funktionaltransformationen. Arch. Math. **9** (1958), 94–101.

For $m=0, 1, 2, \cdots, \infty$, let C_m denote the linear space of real-valued functions f, defined and m times continuously differentiable on the whole real line, and such that f vanishes on some interval of the form $[-\infty, \sigma]$, where σ may depend on f. A linear map $L: C_m \to C_0$ is called "linear transformation on C_m" if L is translation-invariant and, in addition, has the following property: if $f \in C_m$ vanishes on $[-\infty, \sigma]$, then Lf also vanishes on $[-\infty, \sigma]$. Such an L is called continuous of order p ($p=0, 1, 2, \cdots, m$) if the following is true: let f_j ($j=1, 2, \cdots$) be a sequence of functions of C_m vanishing in $[-\infty, \sigma]$ with a fixed σ independent of j and suppose that for each fixed $r=0, 1, 2, \cdots, p$ the rth derivatives $f_j^{(r)}$ of f_j converge uniformly to zero in $[-\infty, \tau]$; then the sequence Lf_j converges uniformly to zero in $[-\infty, \tau]$.

If L on C_∞ is continuous of order ∞ it will, in general, not be continuous of a finite order m. One of the main results of the paper asserts, however, that this is the case if L is subject to restrictions for $t \to \infty$. The precise statement (Satz 4) is as follows: the L of the property just described is continuous of some finite order m if and only if there exists a real-valued monotone increasing function $H(t) \geq 1$ defined for $0 \leq t < \infty$ satisfying

$$(1) \qquad Lf(t) = O(H(t)) \quad \text{for} \quad t \to \infty$$

for all $f \in C_\infty$ vanishing outside a bounded interval $[0, \tau]$. Under the condition stated L is then defined and continuous of order m on a proper subset (i.e., C_∞) of C_m. According to a result contained in a joint paper of the author and J. Meixner [Math. Nachr. **19** (1958), 265–322] L has a unique extension to C_m which is continuous of order m and preserves the norm $\|L\|_m(t)$ defined below. In the present paper it is proved (Satz 2) that (1) holds also for this extension for all $f \in C_m$ vanishing outside a bounded interval $[0, \tau]$. Again, let L be a linear transformation on C_m with m finite. If then (1) holds for all $f \in C_m$ vanishing outside a finite interval $[0, \tau]$, we have

$$\|L\|_m(t) = O(t^{m+1}H(t)) \quad \text{for} \quad t \to \infty$$

where the left member is defined as the sup of

$$\max_{\tau \leq t} |Lf(\tau)|$$

as f varies over those $f \in C_m$ for which $f(t)=0$ for $t \leq 0$ and $|f^{(m)}(t)| \leq 1$ for $t \geq 0$.

The results of this paper are generalizations of results contained in the joint paper by the author and J. Meixner mentioned above. *E. H. Rothe* (Ann Arbor, Mich.)

2181:

Yosida, Kôsaku. On the reflexivity of the space of distribution. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 151–155.

Soit $\mathscr{D}$ l'espace des fonctions indéfiniment dérivables à support compact sur R^n, $\mathscr{D}'$ son dual, l'espace des distributions. Le reviewer a montré que $\mathscr{D}$ était réflexif (que $\mathscr{D}'$ avait pour dual $\mathscr{D}$) par des arguments "vectoriels topologiques"; l'auteur en donne ici une démonstration par l'analyse classique, à l'usage des physiciens. Soit L une forme linéaire continue sur $\mathscr{D}'$. On sait que $\mathscr{D}$ est contenu dans $\mathscr{D}'$, et que la convergence dans $\mathscr{D}$ implique la convergence dans $\mathscr{D}'$; donc L définit a fortiori une forme linéaire continue sur $\mathscr{D}$, c.à.d. une distribution S; pour $\varphi \in \mathscr{D}$, on aura $S(\varphi)=L(\varphi)$. Mais, si φ converge vers 0 dans L^2, elle converge vers 0 dans $\mathscr{D}'$, donc $L(\varphi)$ doit converger vers 0, donc S est nécessairement elle-même une fonction appartenant à L^2. Supposons maintenant que $\varphi = D^p\psi$, et que ψ converge vers 0 dans L^2; alors φ converge vers 0 dans $\mathscr{D}'$, donc $L(\varphi)=(-1)^{|p|}D^pS(\psi)$ converge vers 0, donc D^pS aussi appartient à L^2. Ainsi S a toutes ses dérivées successives dans L^2; un calcul simple (par transformation de Fourier) montre qu'elle est alors une fonction indéfiniment dérivable. Mais si φ converge vers 0 uniformément sur tout compact, $L(\varphi)$ doit aussi converger vers 0, ce qui implique que S ait son support compact. Donc $S=\Phi \in D$. Alors $L(\varphi)=S(\varphi)=\Phi(\varphi)=\varphi(\Phi)$ implique par passage à la limite, du fait que $\mathscr{D}$ est dense dans $\mathscr{D}'$, que $L(T)=T(\Phi)$ pour toute $T \in \mathscr{D}'$, donc $\mathscr{D}'$ a bien pour dual $\mathscr{D}$.

L. Schwartz (Paris)

2182:

Dinculeanu, N. Espaces d'Orlicz de champs de vecteurs. III. Opérations linéaires. Studia Math. **17** (1958), 285–293.

In this paper the author gives the explicit form of the continuous linear functionals on the Banach spaces $L_{\mathscr{M}^\Phi}$, introduced by him in part II [see Atti Acad. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 269–275; MR **19**, 1066], where the Young function Φ satisfies the following two conditions: (i) $\lim \Phi'(t)=l<\infty$ if t tends to infinity, and (ii) $\Phi(2u) \leq M \cdot \Phi(u)$ for all $u \geq 0$ and some constant $M>1$. *W. A. J. Luxemburg* (Pasadena, Calif.)

2183:

Rolewicz, S. Some remarks on the spaces $N(L)$ and $N(l)$. Studia Math. **18** (1959), 1–9.

Let $N(t)$ be a non-negative Baire function. By $N(L)$ [resp. $N(l)$] the author denotes the space of all real measurable functions on (0, 1) [resp. the space of all real sequences $x=\{\xi_n\}$] for which $\rho_N(x)=\int_0^1 N(x(\xi))d\xi<\infty$ [resp. $\rho_N(x)=\sum N(\xi_n)<\infty$]. A sequence x_n is called convergent if $\rho_N(x_n-x)$ tends to zero if N tends to infinity. Conditions are given in order that there exist at least one bounded neighbourhood in these spaces. The question about the existence of continuous linear functions is also investigated (think about the case $N(t)=t^p$, $0<p<1$). It

is shown that they exist on the space $N(L)$ if and only if $\liminf N(t)/t > 0$ as t tends to infinity. At the end of the paper examples are given to illustrate the results given.

W. A. J. Luxemburg (Pasadena, Calif.)

2184:

Luthar, Indar S. Uniqueness of the invariant mean on an abelian semigroup. Illinois J. Math. **3** (1959), 28–44.

Let Σ be a semigroup and $m(\Sigma)$ the space of bounded real-valued functions on Σ with $\|x\| = \sup_{\sigma\in\Sigma} |x(\sigma)|$. A mean ν is an element of the conjugate space $m(\Sigma)^*$ satisfying $\|\nu\| = 1$ and $\nu(e) = 1$, where e is the function which is identically 1.

For a fixed $\sigma \in \Sigma$ let l_σ be the operator carrying $m(\Sigma)$ into itself defined by $(l_\sigma x)(\tau) = x(\sigma\tau)$ for each $x \in m(\Sigma)$ and $\tau \in \Sigma$. A mean ν is called left-invariant if $\nu(l_\sigma x) = \nu(x)$ for all $x \in m(\Sigma)$ and all $\sigma \in \Sigma$. Right-invariant and invariant means are defined in an obvious way.

It is known that every abelian semigroup has an invariant mean. The main result of this paper is the following remarkable theorem: An abelian semigroup Σ has a unique invariant mean if and only if Σ contains a finite ideal. In the case that Σ is an abelian group this reduces to a result proved by M. M. Day [same J. **1** (1957), 509–544; MR **19**, 1067].

In addition to this main theorem the author obtains a number of other partly new results by developing the ideas of M. M. Day.

Št. Schwarz (Bratislava)

2185:

McArthur, Charles W. On totally bounded subsets of sequence spaces. Proc. Amer. Math. Soc. **9** (1958), 881–885.

Cohen and Dunford proved that if X is a Banach space admitting a biorthogonal basis $\{x_i\}$, $\{f_i\}$, then a bounded subset S of X is totally bounded if and only if

$$\lim_n \sum_{i=1}^{n} f_i(x)x_i = x$$

uniformly for $x \in S$ [Duke Math. J. **3** (1937), 689–701]. It is well known that a real Banach space with an orthogonal basis is linearly homeomorphic to a certain Banach space of sequences of real numbers. The main theorem of the note under review is a generalization of the theorem of Cohen and Dunford to certain (not necessarily real) sequence spaces.

Let X be a real Banach space; then a real Banach space W of sequences in X is called an A-space of X if addition and scalar multiplication are as usual for a sequence space and if it satisfies the following four conditions. (i) All finitely non-zero sequences in X are in W. (ii) Let T_n be an operator on W into itself defined as $T_n(s)(i) = s(i)$ if $i \leq n$ and $T_n(s)(i) = 0$ if $i > n$. Then the family $\{T_n : n = 1, 2, \cdots\}$ is uniformly bounded. (iii) $\mathrm{Lim}_n T_n(s) = s$ for each $s \in W$. (iv) For each n and each $s \in W$, $\|K_n(s)\|_W = \|s(n)\|_X$, where $K_n(s)$ is an element of W defined as $K_n(s)(i) = 0$ if $i \neq n$ and $K_n(s)(n) = s(n)$.

The main theorem states that, if W is an A-space of a real Banach space X, then a subset S of W is totally bounded if and only if $\lim_n T_n(s) = s$ uniformly on S and the set $\{s(i) : s \in S\}$ is totally bounded in X for each i. There are several immediate corollaries.

I. Namioka (Ithaca, N.Y.)

2186:

Mihlin, S. G. Two theorems on regularizers. Dokl. Akad. Nauk SSSR **125** (1959), 737–739. (Russian)

Let E_1, E_2 be Banach spaces, $A : E_1 \to E_2$ a closed, linear operator. A bounded operator $B : E_2 \to E_1$ is a regularizer of A if $BA = 1 + T$ where T is compact. The regularizer is said to be equivalent if the equation $Af = g$ follows from $BAf = Bg$.

Theorem 1: A necessary condition that A possess a regularizer is that A be normally solvable in the sense of Hausdorff, i.e., in the present context, that it have a closed range. Theorem 2: Necessary and sufficient conditions that A possess an equivalent regularizer are that A be normally solvable and that n, n^* both be finite with $n \geq n^*$, where n, n^* are the linear dimensions of the null spaces of A, A^*, respectively. Proofs of both theorems are elementary and are given in full. Various special cases of both results have been obtained previously by the author. See, e.g., same Dokl. **57** (1947), 11–12 and Vestnik Leningrad. Univ. **11** (1956), no. 1, 3–24 [MR **9**, 241; **19**, 42].

A. Brown (Houston, Tex.)

2187:

Straus, A. V. Spectral functions of a differential operator. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 185–191. (Russian)

The theory of generalised resolvents and general (nonorthogonal) spectral functions of a nonselfadjoint symmetric operator is applied to the operator A in $L^2(0, 1)$ which is equal to id/dx for functions which are zero at 0 and 1 and have derivative in $L^2(0, 1)$. Any spectral function corresponds to a function θ which is regular in the upper halfplane: if $M(\lambda)$ is given by

$$2[1 - e^{i\lambda}\theta(\lambda)]M(\lambda) = i(1 + e^{i\lambda}\theta(\lambda)),$$

and

$$\pi\rho(t) = \lim_{\tau \to +0} \int_0^t \mathrm{Im}\, M(\sigma + i\tau)d\sigma,$$

then the general form of a spectral function is

$$E_{\alpha,\beta} f = \int_\alpha^\beta d\rho(t) \int_0^1 e^{it(s-x)} f(s)ds.$$

If $|\theta(\lambda)| < 1$ for $\mathrm{Im}\ \lambda > 0$, $|\theta(\lambda)| = 1$ for λ real, the formula gives a series expansion for elements f of Hilbert space in terms of the functions $e^{-i\lambda_n x}$, where $\{\lambda_n\}$ is the set of zeroes of $e^{i\lambda}\theta(\lambda) - 1$.

J. L. B. Cooper (Cardiff)

2188:

Sz.-Nagy, Béla; et Foiaş, Ciprian. Sur les contractions de l'espace de Hilbert. III. Acta Sci. Math. Szeged **19** (1958), 26–45.

The main results concern functional calculus for contraction operators T ($\|T\| \leq 1$) on Hilbert space H. By means of the unitary dilation $U = \int e^{it}dE_t$ of T, which is an essentially unique unitary operator on a space $K \supset H \ni T^n = P_H U^n$ [B. Sz.-Nagy, Acta Sci. Math. Szeged **15** (1953), 87–92; MR **15**, 326; see also same Acta **15** (1954), 104–114; MR **15**, 719] it is possible to define $u(T)$ as $P_H u(U)$, for $u \in L_\infty(E_t)$, and it is shown that $u \to u(T)$ is an algebra homomorphism on the set $\mathcal{O}_T(S_0)$ of functions analytic inside $S_0 = \{|z| \leq 1\}$ and having radial boundary values except on a countable set containing no

eigenvalue of T, that $\|u(T)\| \leq \sup\{|u(\lambda)|;\ |\lambda|<1\}$ for $u \in \mathcal{O}_T(S_0)$, that pointwise convergence (except on a countable subset of ∂S_0 containing no eigenvalue of T) of uniformly bounded sequences $u_n \in \mathcal{O}_T(S_0)$ implies strong convergence of $u_n(T)$, and that the above definition of $u(T)$ is the ordinary one when it applies if T is normal. {Essentially the same results, in a slightly stronger form, were obtained independently by the reviewer [Trans. Amer. Math. Soc. **87** (1958), 108–118; MR **20** #6040].} These and some related results are applied to semi-groups $\{T_t\}$ of contractions, as follows. If $A=(dT_t/dt)_0$ is the generator, the cogenerator T is defined as the "Cayley transform" $T=(A+I)(A-I)^{-1}$ of A and it is shown that T_t is, for each t, respectively normal, unitary, self-adjoint, isometric, if and only if T is the same, and also that the cogenerator of the dilation of $\{T_t\}$ is the dilation of T. An earlier result of Sz.-Nagy [same Acta **18** (1957), 1–14; MR **19**, 873], on the characterization of the dilation of $\{T_t\}$, is here phrased in terms of the cogenerator. The paper concludes with an application of conformal mapping to show that if T is an arbitrary (bounded) operator and S is a bounded spectral set (in the sense of v. Neumann [Math. Nachr. **4** (1951), 258–281; MR **13**, 254]) with ∂S = simple closed curve, then there is an essentially unique normal N with spectrum $(N) \subset \partial S$, acting on a space $K \supset H \ni T^n = P_H N^n$, and the results on functional calculus carry over. One consequence (footnote 22, p. 42) is that if $\lambda_1 \neq \lambda_2$, $\lambda_i \in \partial S$ are eigenvalues of T then their eigenspaces are orthogonal. *M. Schreiber* (Ithaca, N.Y.)

2189:

Jörgens, Konrad. An asymptotic expansion in the theory of neutron transport. Comm. Pure Appl. Math. **11** (1958), 219–242.

G. M. Wing and the reviewer have discussed the transport of neutrons in an infinite slab under the assumption that all neutrons have one speed [Comm. Pure Appl. Math. **8** (1955), 217–234; Duke Math. J. **23** (1956), 125–142; MR **16**, 1120; **18**, 50]. In the present paper the author treats the case of diffusion in a general bounded domain D and permits an arbitrary distribution V of neutron speeds (with some mild restrictions). This is, therefore, a very general problem to which the author contributes substantially.

Writing the transport equation in the form (1) $\partial\psi/\partial t = A\psi$, where A is a certain linear operator, the author defines a Banach space $\mathfrak{B}$ of functions over the domain $D \times V$ such that A is an operator on $\mathfrak{B}$ with dense domain $\mathfrak{A}$. The functions in $\mathfrak{A}$ must satisfy the boundary condition that no neutrons can enter the domain D from outside. He proves that there exists a unique one-parameter semi-group of bounded linear operators $E(t)$ in $\mathfrak{B}$ such that $\psi(t)=E(t)\psi_0$ for $\psi_0 \in \mathfrak{A}$ $(0 \leq t < \infty)$ and $\psi(t)$ satisfies (1), the boundary condition, and the initial condition: strong $\lim_{t\to 0} \psi(t) = \psi_0$. The semi-group $\{E(t)\}$, therefore, constitutes the solution of the initial-value problem.

Furthermore, $E(t)$ is proved to be completely continuous for $t > t_0 > 0$, a fact which implies that A has a discrete spectrum and finite-dimensional generalized eigenspaces. (In the papers of Wing and the reviewer mentioned above, it was found that A contains a continuous spectrum.) It is shown that these conclusions apply in the important case of inelastic scattering and in the "many-velocity-group" case. An upper bound is given for t_0: $t_0 \leq 3\tau$, where τ is the "maximum escape time" of the neutrons in D, i.e., the largest diameter of D divided by the smallest speed. In several cases which are important physically, it is proved that A possesses at least one eigenvalue.

Finally, an asymptotic expansion is given for the solution operator $E(t)$ in the generalized eigenfunctions.

J. Lehner (East Lansing, Mich.)

2190:

Trotter, H. F. Approximation of semi-groups of operators. Pacific J. Math. **8** (1958), 887–919.

This paper represents an attempt to justify many of the approximations of continuous phenomena by discrete. It does this from a function theoretic background. Let X be a Banach space. Then a sequence $\{X_n\}$ of Banach spaces together with a sequence of linear maps $P_n: X \to X_n$ is called an approximating sequence of Banach spaces if $\|P_n\| \leq 1$ and $\lim \|P_n f\| = \|f\|$ for all f in X. A sequence $\{f_n\}$ with f_n in X_n is said to converge to f in X if $\lim \|f_n - P_n f\| = 0$. Finally, if for each n, A_n is an operator on X_n, then $A = \lim A_n$ is defined to be the operator on X given by $Af = \lim A_n P_n f$, its domain being the set of all f in X for which the limit exists.

Now suppose that $\{T_n(t)\}$ is a sequence of semi-groups whose norms are uniformly bounded in both n and t. Let $\{\Omega_n\}$ be the sequence of corresponding infinitesimal generators and set $\Omega = \lim \Omega_n$. The main theorem of the paper is that if Ω has a dense domain and if for some positive λ the range of $\lambda - \Omega$ is dense in X, then the closure of Ω is the infinitesimal generator of a semi-group $T(t)$ and $T(t) = \lim T_n(t)$.

As an application of these notions, the author gives a very illuminating account of the approximation to one-dimensional diffusion processes by random walks.

G. Hufford (Seattle, Wash.)

2191:

Blattner, Robert J. Automorphic group representations. Pacific J. Math. **8** (1958), 665–677.

Certain representations of groups as *-automorphisms of rings of operators are investigated. More particularly, the author is interested in finding conditions on the group, representation and ring which guarantee the existence of outer automorphisms of the ring. As an application the following is shown. Let G be a non-compact simple Lie group, $\mathfrak{A}$ a finite ring of operators and ρ a non-trivial weakly continuous representation of G by automorphisms of $\mathfrak{A}$. Then $\rho(g)$ is inner if and only if g is central.

F. I. Mautner (Paris)

2192:

Rodnyanskiĭ, A. M. On completely continuous vector fields in Banach space. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 556–559. (Russian)

The author announces, without proof, a long list of extensions to Banach spaces of results recently established for Euclidean spaces [Rodnyanskiĭ, Mat. Sb. (N.S.) **37** (79) (1955), 69–82; MR **17**, 288] together with some results which are new even in Euclidean spaces.

L. C. Young (Madison, Wis.)

2193:

Slugin, S. N. A modification of the abstract analogue

to **Chaplygin's method.** Dokl. Akad. Nauk SSSR **120** (1958), 256–258. (Russian)

Chaplygin's method refers to a method of successive approximations which gives a monotonic sequence which approaches the solution from below and a similar sequence of approximations which approaches the solution from above. A number of algorithms of this type have been proposed but are often difficult to apply. This paper presents a "simplified" algorithm. Let P be an operator mapping X into Y, where X and Y are K-spaces. An operator Γ is called "positive-invertable" if it has an inverse $\Gamma^{-1} > 0$. Operators Γ and Λ are called, respectively, the "majorant" and the "minorant" on $[a, b]$ of the increment ΔP if

$$\Gamma(\Delta x) \geq P(x+\Delta x) - P(x) \geq \Lambda(\Delta x),$$

where $[x, x+\Delta x] \subset [a, b]$. I is the identity operator. The principal result is: Assume that on the interval $[a, b] \subset [x_0, \bar{x}_0]$ there exists an additive, positive-invertable majorant $\Gamma(a, b)$ of the increment ΔP; where

$$\Gamma(a, b) \leqq \Gamma(x_0, \bar{x}_0) = \Gamma_0,$$

the operators Γ_0^{-1} and P are monotone continuous, and $P(x_0) \leqq 0 \leqq P(\bar{x}_0)$. Then the algorithm

$$x_{n+1} = x_n - \Gamma_n^{-1}(z_n), \quad \bar{x}_{n+1} = \bar{x}_n - \Gamma_n^{-1}(\bar{z}_n),$$

where $z_n, \bar{z}_n$ are arbitrary elements satisfying $P(x_n) \leqq z_n \leqq 0 \leqq \bar{z}_n \leqq P(\bar{x}_n)$, and $\Gamma(x_n, \bar{x}_n) = \Gamma_n$, defines a sequence such that $x_n \leqq x_{n+1} \leqq x \leqq \bar{x} \leqq \bar{x}_{n+1} \leqq \bar{x}_n$, where $x, \bar{x}$ are the smallest and largest solutions of $P(x) = 0$. If $z_n = \underline{H}_n(P(x_n))$, $\bar{z}_n = \bar{H}_n(P(\bar{x}_n))$ and if there exists a monotonic continuous positive operator $H \leq \underline{H}_n, H \leq \bar{H}_n$, then $x_n \uparrow x$ and $\bar{x}_n \downarrow \bar{x}$. If there exists an additive positive-invertable minorant Λ of the increment ΔP, then the solution is unique. The special case obtained by replacing P by $\Gamma_0^{-1}P$ and by placing $\Gamma_n = I$ is noted. The method is illustrated by a two-point boundary value problem, by solving an equation of the form $f(t_1, t_2, t_3) = 0$, and by the problem of solving $y'(t) - \frac{1}{2}y^3(\frac{1}{2}t) - 1 = 0$, $y(0) = 0$, $0 \leq t \leq 1$.

J. P. LaSalle (Baltimore, Md.)

CALCULUS OF VARIATIONS

See 2477.

GEOMETRY

See also 2019, 2209, 2279.

2194:

Dubikajtis, L. Un système d'axiomes communs à quelques géométries. Ann. Polon. Math. **5** (1958/59), 209–236.

The system of axioms A_0, A_1, A_2, based on those of M. Esser [Duke Math. J. **18** (1951), 475–479; MR **13**, 59; also Dubikajtis, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 423–427; MR **20** #4205], yield the 'universal geometry' $G_0 = G_0(m, p, q)$, which is self-dual if $p = q$. Most of the paper is devoted to deducing the usual incidence relations and the lattice property of G_0 for which models are obtained by assigning to the parameters m, p, q the values: $n+1, 0, 0$ in n-dimensional projective geometry; $n+1, 1, 0$ in affine geometry; $n+2, 2, 0$ in Möbius geometry; $n+2, 1, 2$ in Laguerre geometry; and $n+3, 2, 2$ in Lie geometry. The axioms are shown to be independent and relations between the different models are studied. Further axioms must be added to make G_0 categorical. *G. de B. Robinson* (Toronto, Ont.)

2195:

Jankowski, A. The system of axioms of the Möbius geometry. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 489–494.

Adding to the axioms A_0, A_1, A_2 of the preceding paper, the author outlines a proof that the resulting system is categorical for n-dimensional Möbius geometry.

G. de B. Robinson (Toronto, Ont.)

2196:

Panella, Gianfranco. Lo S_4 lineare affine come piano sopra un quasicorpo, in relazione alla proposizione configurazionale T_2'. Riv. Mat. Univ. Parma **8** (1957), 207–214.

Let γ denote a (commutative) field without a square-root of -1, let S_2 denote the affine plane over γ, and let Γ = the set of all points (x, y) in S_2. The author shows that the elements of Γ form a quasifield (of Veblen and Wedderburn) under the operations $\oplus$ and $*$, where

$$(b_1, b_2) \oplus (a_1, a_2) = (b_1 + a_1, b_2 + a_2),$$

$$(b_1, b_2) * (a_1, a_2) = \begin{cases} (a_1 b_1 - (1 + a_1^2) a_2^{-1} b_2, a_2 b_1 - a_1 b_2), & \text{if } a_2 \neq 0; \\ (a_1 b_1, a_1 b_2), & \text{if } a_2 = 0. \end{cases}$$

The plane Σ_2^* over Γ satisfies, of course, the Little Theorem of Desargues from the ideal line; and its points may be identified with those of the linear affine four-space over γ.

A quadrilateral A_1, A_2, A_3, A_4, of an incidence plane π is said to "support the configuration T_2'" if and only if: its diagonal points D_1, D_2, D_3 are non-collinear, while the quadrilateral having as vertices D_1, D_2, D_3, and one of the points A_i ($i = 1, 2, 3, 4$), has collinear diagonal points. The author proves that, if γ has characteristic 3, every point of Σ_2^* is the vertex of a quadrilateral which supports configuration T_2'. *W. A. Pierce* (Syracuse, N.Y.)

2197:

Boltjanski, W. Zerlegungsgleichheit ebener Polygone. Bul. Inst. Politehn. Iaşi (N.S.) **4 (8)** (1958), 33–38. (Russian and Romanian summaries)

Let $\mathfrak{G}$ be a group of motions in euclidean plane. Two polygons M and M' are $\mathfrak{G}$-equivalent if there exist finite non-overlapping decompositions $M = \bigcup_1^\kappa N_\lambda$ and $M' = \bigcup_1^\kappa N_\lambda'$ such that each polygon N_λ is congruent to N_λ' by a suitable motion in $\mathfrak{G}$; $\lambda = 1, \cdots, \kappa$. $\mathfrak{G}$ is called a $\mathfrak{Z}$-group if two polygons of equal area always are $\mathfrak{G}$-equivalent. By a theorem of Hadwiger and Glur the group $\mathfrak{S}$ of the translations and central symmetries is a $\mathfrak{Z}$-group [Elem. Math. **6** (1951), 97–106; MR **13**, 576]. Hence $\mathfrak{S} \subset \mathfrak{G}$ implies that $\mathfrak{G}$ is a $\mathfrak{Z}$-group. The author proves that conversely every $\mathfrak{Z}$-group contains $\mathfrak{S}$. *P. Scherk* (Boulder, Colo.)

2198:

Cavallaro, Vincenzo G. Angle prismatique. Axes de Steiner. Triangles des centres des carrés. Mathesis **67** (1958), 245–249.

The paper under review is a supplement to an earlier article by the same author [Mathesis **63** (1954), 357–365; MR **16**, 611].

The proofs given in the present paper contain references to other writings of the author where definitions of the terms used may be found. *N. A. Court* (Norman, Okla.)

2199a:

Veldkamp, G. R. The Erdös-Mordell inequality. Nieuw Tijdschr. Wisk. **45** (1957/58), 193–196. (Dutch)

2199b:

Brabant, H. The inequality of Erdös-Mordell again. Nieuw Tijdschr. Wisk. **46** (1958/59), 87. (Dutch)

The Erdös-Mordell inequality states that if O is an arbitrary point within a triangle ABC and P, Q, R are the feet of the perpendiculars from O upon the sides BC, CA, AB respectively, then

$$OA+OB+OC \geqq 2(OP+OQ+OR).$$

In his book *Lagerungen in der Ebene, auf der Kugel und im Raum* [Springer, Berlin-Göttingen-Heidelberg, 1953; MR **15**, 248] Fejes Tóth expressed the desirability of having a purely elementary and simple proof of this result. With this as incentive, a number of such proofs have recently been given. The present notes give four more such proofs.

L. Moser (Edmonton, Alta.)

2200:

Takasu, Tsurusaburo. Parabolic Lie geometry. Yokohama Math. J. **4** (1956), 95–98.

The position of this geometry in the hierarchy of geometry is, in the author's words, as follows: (Non-euclidean geometry):(Euclidean geometry):(equiform geometry)=(dual-conformal geometry):(Laguerre's geometry):(equiform Laguerre's geometry)=(Lie's higher sphere geometry):(X):(Y). Here X is called the P Lie sphere geometry and Y the equiform Lie sphere geometry, these X and Y are the parabolic analogues to Lie higher sphere geometry. Where in equiform space E_{n+2} we have the squared distance of two points, we have in corresponding Laguerre space L_{n+1} the squared tangential distance of two oriented hyperspheres, and in the corresponding parabolic Lie sphere geometry the "parabolic Lie geometrical distance" of two oriented linear hypercomplexes of oriented hyperspheres. If we start with a euclidean E_{n+2} instead of an equiform one, we obtain another (narrower) parabolic Lie geometry.

D. J. Struik (Cambridge, Mass.)

2201:

Brauner, H. Eine Verallgemeinerung der Zyklographie. Arch. Math. **9** (1958), 470–480.

The cone of tangents from a point P to a fixed sphere Φ meets its equatorial plane in a conic C. A detailed study is made of the map $P \to C$, which is made one-to-one by orienting C. In the non-euclidean metric based on Φ as absolute, the conics are oriented circles (cycles), and the map thus is a generalization of the cyclographic map [E. Müller and J. Krames, *Die Zyklographie*, Wien, 1929; see also K. Fladt, Arch. Math. **7** (1956), 391–398; **7** (1957), 399–405; MR **19**, 573]. *F. A. Behrend* (Melbourne)

2202:

Klingenberg, Wilhelm. Eine Charakterisierung von affinen Ebenen mit Orthogonalität. Arch. Math. **9** (1958), 152–154.

If we have an affine plane A and a non-trivial involution t of its line at infinity we may define a relation of orthogonality on pairs of lines of A whenever their points at infinity are interchanged by t. The author assumes this axiom: If P, P^1 and Q, Q^1 are pairs of distinct points on two orthogonal lines, there is a collineation taking P into P^1, Q into Q^1 and inducing the involution t on the line at infinity. This note shows that there is an exact correspondence between planes with the above property and involutions of a one-dimensional projective linear group over a division ring. *Marshall Hall, Jr.* (Columbus, Ohio)

2203:

Szász, Paul. Neuer Beweis für die Darstellung der Bewegungen und Umwendungen der hyperbolischen Ebene mit Hilfe der Hilbertschen Endenrechnung. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 67–70.

The author has given a simple proof [Acta Math. Acad. Sci. Hungar. **4** (1953), 243–250; MR **15**, 645; see especially pp. 246–247] for H. Liebmann's theorem: "Die Bewegungen bzw. Umwendungen der hyperbolischen Ebene sind, auf die Enden ξ bezogen, analytisch durch die lineare Transformationen $\xi' = (\alpha\xi+\beta)/(\gamma\xi+\delta)$ $(\alpha\delta-\beta\gamma>0)$ bzw. $\xi' = (\alpha\xi+\beta)/(\gamma\xi+\delta)$ $(\alpha\delta-\beta\gamma<0)$ dargestellt" [see Liebmann, Math. Ann. **59** (1904), 110–128]. Here he gives a new, intuitive proof, which includes a proof of the following theorem of B. Kerékjártó [Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59** (1940), 19–61; Comment. Math. Helv. **13** (1940), 11–48; MR **2**, 259]: "Jede Bewegung der hyperbolischen Ebene kann aus folgenden Bewegungen zusammengesetzt werden: (a) Halbdrehung um einen gegebenen Punkt O; (b) Verschiebung längs einer Geraden $O\Omega$, wobei Ω ein gegebener Fernpunkt ist; (c) Drehung um den Fernpunkt Ω.

T. Takasu (Yokohama)

CONVEX SETS AND GEOMETRIC INEQUALITIES

2204:

Lenz, Hanfried. Die Eilinien mit einer Schar konjugierter Durchmesserpaare. Arch. Math. **9** (1958), 134–139.

A P-curve [Blaschke, Math. Nachr. **15** (1956), 158–264; MR **18**, 922] is a convex curve with the property that each diameter d has a conjugate d', i.e., a diameter parallel to the support lines at the end points of d. An R-curve (Radon curve) is a P-curve with center of symmetry. The paper gives a way to generate R-curves and some extremal properties of P- and R-curves. For instance, if E_1 is the area of the greatest inscribed quadrilateral and

E_2 that of the smallest circumscribed parallelogram to a convex curve K, the inequality $E_2/E_1 \leq 2$ holds, with equality for P-curves. *L. A. Santaló* (Buenos Aires)

2205:

Wirsing, Eduard. Zur Analytizität von Doppelspeichenkurven. Arch. Math. **9** (1958), 300–307.

An equichordal point of a closed curve is a point which has the property that every straight line through the point meets the curve in exactly two points and the distance between the two points is the same for all lines through the equichordal point. In 1917 Blaschke, Rothe and Weitzenböck raised the question of whether a closed convex curve can have more than one equichordal point [Arch. Math. Phys. **27** (1918), 82; Aufgabe No. 552]. It is known that such a curve can have at most two equichordal points, but whether it can or cannot have two is still undecided. In the present paper it is proved that any closed curve with two equichordal points is analytic regular at all its points. The author indicates that the method used in proving this can be extended to show that there are at most enumerably many such curves which are not similar. *G. A. Dirac* (Hamburg)

2206:

Grünbaum, Branko. On common secants for families of polygons. Riveon Lematematika **12** (1958), 37–40. (Hebrew. English summary)

A family of compact convex sets M in E^n is said to possess property T if there exists a straight line meeting all sets of M. It is said to have property $T(k)$ if every subfamily consisting of k sets has property T. The following result is proved: To every convex polygon P in E^2 there corresponds an integer k such that $T(k)$ implies T for the family of all polygons similar to P and whose sides are parallel to the corresponding ones of P. It is furthermore shown that $k \leq 3\binom{p}{2}$ if P is a p-gon.

A. Dvoretzky (Jerusalem)

2207:

Šeftel', Z. G. On the point deviating least (in the sense of P. L. Čebyšev) from a given finite system of planes and points. Dokl. Akad. Nauk SSSR **125** (1959), 289–292. (Russian)

Consider a finite set $\mathfrak{F}$ of functions $f(x)=a_1x_1+\cdots+a_nx_n+b$ on R^n, a finite set P of points y in R^n, and a positive real function λ on P. The problem considered is that of minimizing the function $\Delta(x)=\max\{|f(x)|: f\in\mathfrak{F};\ \lambda(y)|x-y|: y\in P\}$. An algorithm is given for determining the point where Δ assumes its minimum, which it always does. *E. Hewitt* (Seattle, Wash.)

2208:

Baebler, F. Über die Eindeutigkeit der Lösung des isoperimetrischen Problems. Nachtrag zu einer früheren Publikation. Arch. Math. **8** (1957), 464–468.

The unicity of the isoperimetric problem, which in a previous paper of the author [Arch. Math. **8** (1957), 52–65; MR **19**, 977] was proved by symmetrization, is here proved by a more direct and elementary way.

L. A. Santaló (Buenos Aires)

2209:

Grünbaum, B. Borsuk's partition conjecture in Minkowski planes. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 25–30.

The author shows that any subset K of a Minkowski plane whose unit sphere is not a parallelogram can be covered by three spheres each having a diameter smaller than diameter K. He also makes some remarks concerning the diameter J_s of the smallest sphere which can cover any set of diameter 1 and suggests the conjecture that any set of diameter 1 can be covered by three spheres of diameter $1/J_s$. *H. G. Eggleston* (London)

2210:

Abel, William R.; and Blumenthal, Leonard M. Distance geometry of metric arcs. II. Arch. Math. **8** (1957), 451–463.

[For part I, see Amer. Math. Monthly **64** (1957), no. 8, part II, 1–10; MR **20** #5462.] A metric continuum K of a metric space M is said to have property $T_{\pi/2}{}^*$ at a point p of K if and only if positive numbers δ_p, α_p ($\alpha_p \leq \pi/2$) exist such that each triple of pairwise distinct points of $K\cdot U(p;\delta_p)$ contains an angle greater than or equal to $\pi/2+\alpha_p$, where $U(p;\delta_p)=[q\in M|pq<\delta_p]$. If $\alpha_p=0$ the property is denoted by $T_{\pi/2}$. These properties, which do not involve the notion of limit, are used in order to obtain certain theorems on distance geometry of metric arcs, the principal ones of which seem to be the following: (a) Any metric arc with property $T_{\pi/2}{}^*$ at each of its points is rectifiable; (b) a metric continuum with property $T_{\pi/2}{}^*$ at each of its points is a rectifiable arc or a rectifiable simple closed curve; (c) if a metric continuum K has property $T_{\pi/2}{}^*$ at one of its points p_0, then K is a rectifiable arc in a neighborhood of p_0. Several consequences are deduced; for instance, the proof of the following theorem stated without proof by Paue: If a metric continuum is flat at a point p_0 (i.e., $\lim\max\sphericalangle(r,s,t)=\pi$ as $r,s,t\to p_0$) then in a neighborhood of p_0 it is a rectifiable arc.

L. A. Santaló (Buenos Aires)

GENERAL TOPOLOGY, POINT SET THEORY

See also 2055, 2177, 2192, 2244, 2245.

2211:

Edelstein, Michael. A further generalization of a problem by Sylvester. Riveon Lematematika **11** (1957), 50–55. (Hebrew. English summary)

The theorem of Grünbaum [Riveon Lematematika **10** (1956), 46–48; MR **19**, 667] is that if a finite family $\{S\}$ of compact, connected and mutually disjoint subsets of the plane has the property that no straight line meets exactly two sets S, then the union U of $\{S\}$ is part of a straight line. The author extends this to n-space and furthermore admits sets S with finitely many (for $n=2$, denumerably many) connected components, as long as not all components of U are single points.

T. S. Motzkin (Los Angeles, Calif.)

2212:

Sekanina, Milan. Decompositions of the euclidean spaces. Časopis Pěst. Mat. **83** (1958), 70–79. (Czech. Russian and English summaries)

The paper deals with disjoint partitions of euclidean spaces R_n. For sets M_1, M_2 in R_n the relation $M_1 \simeq M_2$ [resp. $M_1 \doteq M_2$] means that M_1, M_2 are [directly] congruent. A set $S \subseteq R_n$ is said to be a [direct] partition model of $M \subseteq R$ if there exists a disjoint partition $M(x)\,(x \in S)$ of M such that $M(x) \simeq S$ [resp. $M(x) \doteq S$] for every $x \in S$. Th. 1: Let $n > 1$, m a cardinal $\neq 0$ and $< c\,(= 2^{\aleph_0})$, and S a system of c sets $\neq 0$ in R_n, each of cardinality $\leqq m$; then there exists a mapping f of S into R_n (i.e., $fx \subseteq R_n$ for every $x \in S$) such that $fx \doteq x\,(x \in S)$ and that $fx\,(x \in S)$ is a disjoint partition of R_n. For $n = 1$, the statement does not hold; however (Th. 3), for every infinite subset M of R_1 and every $\varepsilon > 0$ there exists a set $A \supseteq M$ satisfying $\operatorname{diam} A \leqq \operatorname{diam} M + \varepsilon$ and such that A is a partition model of R_1; in particular (Th. 4): If A is a direct partition model of R_1 containing the triadic set, then $\operatorname{diam} A > 1$ and $\mu^* A \geqq 1$. Let $n \in \{0, 1, \cdots\}$; to every system S of c non-void sets in R_n there corresponds a disjoint partition $fx\,(x \in S)$ of R_{2n+1} such that $fx \doteq x\,(x \in S)$. The proofs are carried out by transfinite induction. *Đ. Kurepa* (Zagreb)

2213:

Sekanina, Milan. On certain decomposition sets of the plane. Časopis Pěst. Mat. **84** (1959), 74–82. (Czech. Russian and English summaries)

The paper is connected with that reviewed above, and deals in particular with disjoint partitions of the euclidean plane E_2 into sets each congruent with a given set N (partition model). If N is a model of the disjoint partition P of E_2 and if N is located on a line $l \subset E_2$ and $M = l \setminus N$, $m = \operatorname{card} M < c$, let, for every $x \in P$, px denote the line containing x; then through every point of the plane E_2 we have $\leqq \sup\{m, \aleph_0\}$ sample-lines px, and there are 2 lines $p_1, p_2 \subset E_2$ and a partition $\mathfrak{R}_1, \mathfrak{R}_2$ of the set $\mathfrak{R}$ of the lines $px\,(x \in P)$ such that every element of $\mathfrak{R}_i$ is parallel to p_i for $i = 1, 2$ (Th. 1). If $a \in M$, there exists a translation t of l satisfying $tM \subseteq M$, a non $\in tM$ (Th. 2). This condition $m = \aleph_0$ is also sufficient in order that N be a partition model for E_2. *Đ. Kurepa* (Zagreb)

2214:

de Figueiredo, Djairo Guedes. Decompositions of the sphere. Notas Mat. No. 14 (1958), 40 pp.

Expository. *E. Silverman* (Lafayette, Ind.)

2215:

Ward, L. E., Jr. A fixed point theorem for multi-valued functions. Pacific J. Math. **8** (1958), 921–927.

For a topological space Y denote by $S(Y)$ the space of closed subsets of Y endowed with the finite topology. A multi-valued function $F: X \to Y$ is said to be continuous provided $F(x)$ is a closed set for each x in X, and the induced (single-valued) function $f: X \to S(Y)$ is continuous in the usual sense. The space X is said to have the F.p.p. (=fixed point property for continuous multi-valued functions) if and only if for each such $F: X \to X$ there is an x in X such that x belongs to $F(x)$.

Let X be a space and $\leqq$ a partial order on X. We write $L(x) = \{a : a \leqq x\}$, $M(x) = \{a : x \leqq a\}$, $[x, y] = M(x) \cap L(y)$. An antichain of X is a subset in which no two distinct elements are related under the partial order. The author considers the following five properties which are enjoyed by some spaces admitting a partial order, $\leqq$. I. $[x, y]$ is closed and simply ordered for each x and y in X. II. $\leqq$ is order-dense. III. There exists an element e of X such that $M(e) = X$. IV. If x and y are points of the subcontinuum Y and $x \leqq y$, then $[x, y]$ is contained in Y. V. If A is an antichain of X, and P is a continuum contained in the union of all $M(x)$ for which x is in A, then P is contained in $M(x)$ for some x in A.

A continuum (=compact connected Hausdorff space) is topologically chained if each pair of its points is contained in a subcontinuum which has exactly two non-cutpoints.

The theorems established are the following. Theorem 1: A necessary and sufficient condition that X be a topologically chained, hereditarily unicoherent continuum is that X be a compact Hausdorff space which admits a partial order satisfying I–V. Theorem 2: Each topologically chained, hereditarily unicoherent continuum which contains no indecomposable continuum has the F.p.p.

D. W. Hall (Endicott, N.Y.)

2216:

Whyburn, Gordon T. On the invariance of openness. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 464–466.

Étude des espaces pour lesquels l'ouverture est un invariant topologique. Soit X un continu généralisé localement connexe possédant cette propriété (dite de Brouwer), et $Y = f(X)$ un autre continu généralisé localement connexe; s'il existe un ensemble fermé $F \subset Y$ tel qu'aucun ensemble localement compact homéomorphe à un sous-ensemble de F ne sépare aucune région dans Y, et si de plus f est localement topologique sur $X - f^{-1}(F)$, alors Y possède aussi la propriété de Brouwer. L'auteur en déduit la construction de plusieurs exemples d'espaces possédant la propriété de Brouwer sans être des variétés Euclidiennes. *L. Fourès* (Marseille)

2217:

Hayashi, Eiichi. One point expansion of topological spaces. Proc. Japan Acad. **34** (1958), 73–75.

Let $\mathfrak{R}$ be a dual filter in a topological space R. Let $p \notin R$ and $R^* = R \cup \{p\}$. A topology for R^* can be obtained as follows ($X \subset R^*$, $\overline{X}$ and $\overline{\overline{X}}$ denote the closures of X in R and R^* respectively): $\overline{\overline{X}} = \overline{X}$, if $X \subset R$ and $\overline{X} \in \mathfrak{R}$; $\overline{X} \cup \{p\}$, if $X \subset R$ and $\overline{X} \notin \mathfrak{R}$; $\overline{X - \{p\}} \cup \{p\}$ if $p \in X$. R^* contains R as a dense subspace, and is called the one point expansion of R (with respect to $\mathfrak{R}$). Every topological space R has a trivial one point expansion obtained by letting $\mathfrak{R} = \{\emptyset\}$. If $\mathfrak{R}$ is the collection of all subsets of R whose closures are compact, then we have the Alexandroff one point compactification of a non-compact space. The case where $\mathfrak{R}$ is the set of all meager sets in R is also studied.

{The paper contains several errors. Theorem 2 contains hypotheses which cannot be simultaneously satisfied.}

Hing Tong (Middletown, Conn.)

2218:

Dieudonné, J. Un critère de normalité pour les espaces produits. Colloq. Math. **6** (1958), 29–32.

The author first shows that, if E is a paracompact first countable T_2 space and F is normal and countably compact, then $E \times F$ is normal. This theorem is extended to a type of "local" product, and analogous problems for

identification spaces are then considered. Let f be a closed continuous mapping of a T_2 space G onto a space E. If E is paracompact and each set $f^{-1}(x)$, $x \in E$, is compact, it is shown that G is paracompact. If instead E is paracompact and first countable, and each set $f^{-1}(x)$, $x \in E$, is normal and countably compact, it is conjectured that G must be normal. {As the proofs show, one could weaken "first countable", in the first two theorems, to the requirement that each point has a basis of $\leqq m$ neighborhoods, providing countable compactness is strengthened to m-compactness. One can also conclude, in the first two theorems, that the product is countably paracompact.} *A. H. Stone* (Manchester)

2219:

Katetov, M. Extension of locally finite coverings. Colloq. Math. **6** (1958), 145–151. (Russian)

A family $\{X_\alpha\}$ of subsets of topological space P is "uniformly locally finite" if there exists a locally finite open covering $\{U_\beta\}$ of P such that for all β, $X_\alpha \cap U_\beta \neq \emptyset$ for only finitely many α. P is "strongly normal" if P is both collectionwise normal and countably paracompact. The author gives several theorems concerning collectionwise normal and strongly normal spaces, of which the following are representative. (1) $\{X_\alpha\}$ is uniformly locally finite in P if and only if there exists a locally finite open family $\{G_\alpha\}$ such that $X_\alpha^- \subset G_\alpha$. (2) For a normal space P each of the following properties is equivalent to collectionwise normality: (a) If $\{F_\alpha\}$ is a closed locally finite family in P such that for some positive integer n the intersection of any $n+1$ members of $\{F_\alpha\}$ is empty, then there exists an open locally finite family $\{G_\alpha\}$ such that $F_\alpha \subset G_\alpha$. (b) For any closed subset S of P, if $\{F_\alpha\}$ [respectively, $\{H_\alpha\}$] is a closed [respectively, open] locally finite family for the relative topology of S such that $F_\alpha \subset H_\alpha$, then there exists an open locally finite family $\{G_\alpha\}$ in P such that $F_\alpha \subset G_\alpha \cap S \subset H_\alpha$. (c) For any closed subset S of P, if $\{X_\alpha\}$ is a uniformly locally finite family for the relative topology of S, then $\{X_\alpha\}$ is uniformly locally finite in P. (d) For any closed subset S of P, if $\{X_\alpha\}$ is a uniformly locally finite open covering of S for its relative topology, then there exists a uniformly locally finite open covering $\{G_\alpha\}$ of P such that $X_\alpha = G_\alpha \cap S$. (3) If P is normal, then P is strongly normal if and only if every locally finite family of subsets of P is uniformly locally finite.

{Reviewer's remarks. From (1) it follows easily that if $\{X_\alpha\}$ is uniformly locally finite in normal P, then there exists a uniformly locally finite open covering $\{H_\alpha\}$ such that $X_\alpha^- \subset H_\alpha$; this, applied to S, suffices to show (d) implies (c) in (2). The author's proof that (b) implies (d) in (2) does not establish that $\{G_\alpha\}$ is uniformly locally finite. The proof may be completed as follows: By (1) there exists an open, locally finite collection $\{H_\alpha\}$ for the relative topology of S such that $X_\alpha^- \subset H_\alpha$ and by (b) there exists an open locally finite covering $\{U_\alpha\}$ of P such that $X_\alpha^- \subset U_\alpha \cap S \subset H_\alpha$. By a well-known theorem [N. Bourbaki, *Topologie générale*, 2nd ed., ch. 9, p. 87] there exists an open locally finite covering $\{V_\alpha\}$ of P such that $V_\alpha^- \subset U_\alpha$; by normality we may further suppose $X_\alpha^- \subset V_\alpha$. If W_α is an open set in P containing $P - S$ such that $W_\alpha \cap S = X_\alpha$, then the sets $G_\alpha = V_\alpha \cap W_\alpha$ form the desired uniformly locally finite open family in P by (1). Finally, in (5.5) one should assume that each X_α is open for the relative topology of S.} *S. Warner* (Durham, N.C.)

2220:

Obreanu, Filip. Quelques problèmes sur la compactification. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 457–459.

It is shown that a space which is uniformly locally compact in some uniform structure necessarily admits several precompact uniform structures, and hence several compactifications. *R. Arens* (Los Angeles, Calif.)

2221:

Nollet, Louis. A propos de la locale compacité des espaces topologiques. Bull. Soc. Roy. Sci. Liège **28** (1959), 5–13.

The author considers three definitions of local compactness for a space X, observing that they are equivalent when X is Hausdorff or regular or hereditarily compact, but not in general. {His distinction between "ouvertement compact", "complètement compact" and the property that every strictly decreasing sequence of closed sets is finite, is unnecessary, as these properties are equivalent. His counterexamples are not T_1, but could be replaced by others which are.} He also considers local finiteness; the two natural definitions are always equivalent, and he obtains a particular case of the following result. Given a system $\{P_\alpha\}$ of subsets of a space X, let F denote any finite set of suffixes α, and write $G_F = X - \bigcup \{\overline{P}_\alpha | \alpha \notin F\}$; then $\{P_\alpha\}$ is locally finite if and only if $\{G_F\}$ is an open covering of X. *A. H. Stone* (Manchester)

2222:

Fadell, Edward. B paracompact does not imply B^I paracompact. Proc. Amer. Math. Soc. **9** (1958), 839–840.

The assertion in the title is proved by means of a suitable example. *E. Michael* (Seattle, Wash.)

2223:

Harrold, O. G., Jr. A sufficient condition that a monotone image of the three-sphere be a topological three-sphere. Proc. Amer. Math. Soc. **9** (1958), 846–850.

Let S^3 denote the three-sphere. The author proves the following theorem. Let $M = f(S^3)$, where f is a monotone, continuous mapping such that if Y is the set of all points in M having nondegenerate inverses under f, y is any point in the closure of Y, and $\varepsilon > 0$ is any real number, then there is a topological 2-sphere K in $S(y, \varepsilon)$ separating y and $M - S(y, \varepsilon)$ such that K does not intersect the closure of Y. Then M is a topological 3-sphere.
D. W. Hall (Endicott, N.Y.)

ALGEBRAIC TOPOLOGY

See also 2508a-b.

2224:

Markov, A. Unsolvability of certain problems in topology. Dokl. Akad. Nauk SSSR **123** (1958), 978–980. (Russian)

Let $\mathfrak{R}$ be a binary relation between complexes; e.g., $\mathfrak{R}$ may be the relation of combinatorial equivalence as defined by Alexander [Ann. of Math. (2) **31** (1930), 292–320]. The general decision problem for $\mathfrak{R}$ is the problem of

finding an algorithm which will decide for any two complexes whether $\mathfrak{R}$ holds between them. Certain special decision problems for $\mathfrak{R}$ are obtained by fixing one of the complexes, or by fixing the dimension or the highest dimension of the complexes involved, or by combinations of these restrictions. The present article shows that for a rather wide class of binary relations between complexes (including combinatorial equivalence) these decision problems are unsolvable. The class is described as follows. If K and L are two complexes, K is said to be related to L if K and L are both connected and their fundamental groups are isomorphic, or if either K or L is not connected. Thus relatedness is a binary relation between complexes. Let $\mathfrak{R}$, $\mathfrak{S}$, and $\mathfrak{T}$ be binary relations between complexes. $\mathfrak{R}$ is said to be stronger than $\mathfrak{S}$ if whenever $\mathfrak{R}$ holds $\mathfrak{S}$ holds, and $\mathfrak{S}$ is said to be between $\mathfrak{R}$ and $\mathfrak{T}$ if $\mathfrak{R}$ is stronger than $\mathfrak{S}$ and $\mathfrak{S}$ is stronger than $\mathfrak{T}$. The principal result of the paper is contained in the theorem: For $n > 3$ one can exhibit a connected, closed n-dimensional manifold M^n such that if $\mathfrak{R}$ is any binary relation between combinatorial equivalence and relatedness, then the decision problem for $\mathfrak{R}$ with respect to M^n among n-dimensional manifolds is unsolvable. A proof of this theorem parallels the proof of theorem 1 of Markov [same Dokl. **121** (1958), 218–220; MR **20** #4260] which states the unsolvability of the homeomorphism problems for certain manifolds M^n where $n > 3$. The symbol $\mathfrak{M}(P_1{}^* \cdots {}^*P_m{}^*r)$ of the earlier paper now denotes a complex rather than a polyhedron; the lemmas, with 'homeomorphic' replaced by 'combinatorially equivalent', hold. By these lemmas the decision problem for $\mathfrak{R}$ is reduced to the triviality problem for (r, k)-groups, which was shown unsolvable by Adyan [same Dokl. **103** (1955), 533–535; MR **18**, 455] and Rabin [Ann. of Math. (2) **67** (1958), 172–194].

W. J. Feeney (Weston, Mass.)

2225:

Ulčar, Jože. Über die Streckenkomplexe. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **9** (1956), 1–13. (Serbo-Croatian. German summary)

Die Arbeit schliesst sich an eine vorhergehende Arbeit des Verf. an [dieselbe Annuaire **8** (1955), 3–25; MR **17**, 1232; auch Kurepa, Bull. Soc. Math. Phys. Serbie **5** (1953), no. 3–4, 23–31; C. R. Acad. Sci. Paris **237** (1953), 1133–1135; MR **16**, 227; **17**, 467]. Sei K ein Streckenkomplex im R^3, also $K = S_1 \cup S_0 \cup S_0'$, wo S_1 ein $\leq \aleph_0$ System paarweise teilerfremden Strecken bedeutet; S_0 bedeute die Menge der Endpunkte von Elementen aus S_1; S_0' bedeute eine $\leq \aleph_0$ Menge aus $R^3 \setminus \bigcup S_1$. K_v bedeute einen K, dessen Zusammenhangszahl gleich v ist. M_K bedeute die umfangsreichste und die zahlreichste Untermenge von K, die aus paarweise nicht-inzidenten Elementen besteht. Das Ziel der Arbeit ist die Menge M_K zu bestimmen. Die Menge M_{K_1} besteht aus der Menge aller Ecken von K_1, und aus Mengen, deren jede alle 1-Seiten eines beliebigen zusammenhängenden Unterkomplexes K_1' von K_1 dessen Zusammenhangszahl 1 ist und dazu alle 0-Seiten von $K_1 \setminus \bigcup K_1'$ enthält. Ist $v > 1$, so sind die Elemente von M_{K_v} der Form $M_1 \cup M_0$; dabei bedeutet M_1 die Menge aller 1-Seiten eines beliebigen Unterkomplexes K_v' von K_v, mit der Zusammenhangszahl v, M_0 ist die Menge aller 0-Seiten von $K_v \setminus \bigcup K_v'$. Antwortend auf eine Frage aus seiner zitierten Arbeit findet Verf. die Bedingung, unter welcher die Menge M_K aus einem einzigen homogenen Element besteht. Dieser Fall tritt ein, wenn $S_0' = \emptyset$ (leer) und K keinen Rand hat oder falls die Zusammenhangszahl jeder 1-dim. Komponente von K Null ist.

Đ. Kurepa (Zagreb)

2226:

Mardešić, Sibe. On inverse limits of compact spaces. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **13** (1958), 249–255. (Serbo-Croatian summary)

Let $\{X_\alpha; \pi_{\beta\alpha}\}$ be an inverse spectrum of compact Hausdorff spaces, and let $X = \varprojlim X_\alpha$. The author introduces a topology in the disjoint union of sets $X^* = X \cup \bigcup_\alpha X_\alpha$, making X^* into a paracompact Hausdorff space, such that each X, X_α has its original topology, and such that the directed sequence of subspaces $\{X_\alpha\}$ converges to X. This space is used in the proof of his main theorem: If R is a metric ANR for metric spaces, and the X_α are Hausdorff compacta, then for singular homology and any abelian group G, $\varinjlim H_n(R^{X_\alpha}; G)$ is naturally isomorphic to $H_n(R^X; G)$ (the compact-open topology is used in the function spaces).

J. Dugundji (Los Angeles, Calif.)

2227:

Massey, W. S. On the universal coefficient theorem of Eilenberg and MacLane. Bol. Soc. Mat. Mexicana (2) **3** (1958), 1–12.

The author gives a new proof of the well-known universal coefficient theorem in a more general setting. The exact sequence

$$0 \to \mathrm{Ext}\,[H_{q-1}(X), G] \xrightarrow{\Psi} H^q(X; G) \xrightarrow{\Omega} \mathrm{Hom}\,[H_q(X), G] \to 0$$

(relating the singular cohomology groups $H^*(X; G)$ of a space X with coefficient group G to the integral singular homology group $H_*(X)$) appears as a special case (where G has degree 0) of a more general result concerned with homogeneous homorphisms and graded extensions for a pair of graded differential groups A and B. A homomorphism $f: A \to B$ is called homogeneous of degree q if $f(A_n) \subset B_{n+q}$ for all n; $\hom_q(A, B)$ is the group of homogeneous homomorphisms of degree q, and the graded group $\hom(A, B)$ is the direct sum of $\hom_q(A, B)$ for all q.—An extension (X, φ, ψ) associated with the exact sequence $0 \to B \xrightarrow{\varphi} X \xrightarrow{\psi} A \to 0$ is a graded extension (over A with kernel B) of degree n if X is a graded group and φ and ψ are homogeneous homomorphisms, with $n = -(\text{degree } \varphi + \text{degree } \psi)$; $\mathrm{ext}_n(A, B)$ is the abelian group of extensions of degree n (identified with the cartesian product of $\mathrm{Ext}(A_q, B_{q+n})$ for all q); $\mathrm{ext}(A, B)$ is the direct sum of $\mathrm{ext}_n(A, B)$ for all n.

A and B being differential groups, it is possible to define $H(\hom(A, B))$ and a natural map $\Omega: H(\hom(A, B)) \to \hom(H(A), H(B))$, which is a homomorphism onto if A is free. The isomorphism Ξ (or Ψ^{-1}) of the kernel $\Omega^{-1}(0)$ onto $\mathrm{ext}(H(A), H(B))$ is defined using the algebraic mapping cylinder [also described by S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, 1952; MR **14**, 398; p. 159, exercise D] $M(f)$, which is a differential graded group associated with the homogeneous homomorphism $f: A \to B$. When f is such that $f_* \in \Omega^{-1}(0)$, the exact sequence of the mapping

cylinder $M(f)$ becomes an exact sequence of length three

$$0 \to H(B) \xrightarrow{i_*} H(M(f)) \xrightarrow{j_*} H(A) \to 0$$

defining the graded extension $\Xi(f_*) = (H(M(f)), j_*, i_*)$; the sequence

$$0 \to \operatorname{ext}(H(A), H(B)) \xrightarrow{\nabla} H(\hom(A, B)) \xrightarrow{\Omega} \hom(H(A), H(B)) \to 0$$

is exact.

{On page 3, line 16, the equality sign in "$M(f) = A \times A'$, with the graded structure . . ." should be replaced by inclusion "$\subset$".} *G. Hirsch* (Brussels)

2228:

Yokota, Ichiro. On the homology of classical Lie groups. J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 93–120.

Continuing the work of C. E. Miller [Ann of Math. (2) **57** (1953), 90–114; MR **14**, 673] and his own work [J. Inst. Polytech Osaka City Univ. Ser. A **7** (1956), 39–49; MR **18**, 918], the author gives cell decompositions of the classical groups and some related spaces (Stiefel manifolds, $SO(2n)/U(n)$, $SU(2n)/Sp(n)$), and derives from this the homology and cohomology, cup products, Pontryagin ring, and, to some extent, Steenrod powers. The cell decompositions are the following type: G is transitive on the sphere S_n, with stability group H; e_n is a suitable n-cell that represents a generator of $\pi_n(G, H)$; there is a cell complex on H, which together with the products of the cells by e_n yields a cell complex for G. The e_n's utilized are essentially the familiar ones [cf. N. E. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, 1951; MR **12**, 522; pp. 119, 125, 130]. For the description of the mappings and the tabulation of the results we have to refer to the paper itself. Many of the results on the homology are of course familiar; it is not made very clear which ones are new. {The argument p. 111, l. 16 ff. is unclear; it is certainly not true that $x * x = 0$ for every x.} An appendix shows that some attending maps of the cells in low dimensions are related to the Hopf map and the Blakers-Massey map.

H. Samelson (Ann Arbor, Mich.)

2229:

Thomas, Emery. The functional Pontrjagin cohomology operations. Bol. Soc. Mat. Mexicana (2) **3** (1958), 19–24.

An example is given of non-trivial functional operations associated with the generalized Pontrjagin cohomology operations which the author defined [Mem. Amer. Math. Soc. no. 27 (1957), MR **20** #5471]. The mapping is the fibre map of the sphere S^{2p-1} onto the complex projective space M_{p-1} of $p-1$ complex dimensions; the functional operation associated with the Pontrjagin power $\mathscr{P}_p$ of the generator of $H^2(M_{p-1}; Z)$ generates the cohomology group of the sphere (which can also be obtained by the functional cup-product). A slight modification of this example (with a 3-cell attached to S^2 in M_{p-1} by a suitable map) gives another example, where the functional operation associated with the Pontrjagin power yields new information and shows that the map of S^{2p-1} into this $(2p-2)$-space is essential, whereas the functional cup-product fails to do so.

G. Hirsch (Brussels)

2230:

Yo, Ging-tzung. The construction for Steenrod's D_i in reduced powers of cohomology classes. Acta Math. Sinica **8** (1958), 369–383. (Chinese. English summary)

Steenrod's cyclic reduced power operations St_p^i for a regular cell complex are defined by means of a set of homomorphisms $D_i : C_q(K) \to C_{q+i}(K^p)$, where K^p denotes the p-fold product of K. These homomorphisms must satisfy certain well-known conditions [see N. E. Steenrod, Ann. of Math. (2) **56** (1952), 47–67; MR **13**, 966]. As is well known, the operation St_2^i coincides with the squaring operation Sq^i, which can be defined by means of cup-i products in an ordered simplicial complex. The purpose of this paper is to give a concrete definition for the homomorphisms D_i which will give the cup-i products in the special case $p = 2$.

As is to be expected, most of the paper consists of lengthy formulas and algebraic computations.

W. S. Massey (Providence, R.I.)

2231:

★Adem, José. Second order cohomology operations associated with Steenrod squares. (Spanish.) Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 186–221. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

The paper describes secondary cohomology operations associated with the relations between iterated Steenrod squares which the author proved in an earlier paper [*Algebraic geometry and topology, A symposium in honor of S. Lefschetz*, pp. 191–238, Princeton Univ. Press, 1957; MR **19** 50]; essentially the same method is used here, of which an outline is given (with less details than in the former paper) in the first part of this paper. Iterated squares appear as homology classes of the 2-Sylow subgroup of the symmetric group S_4; however, the canonical π-free acyclic complex V (where π is the cyclic group of order 2) is now replaced by a complex U with augmentation. The relations between iterated squares result from the comparison of two different mappings (depending on an interior automorphism of S_4) of $U \otimes U$ in an S_4-free acyclic S_4-complex W, thus defining an equivariant homotopy. This is applied to the usual construction (using the tensor product with the π-free acyclic complex V) that leads to the definition of the Steenrod squares [N. E. Steenrod, Comment. Math. Helv. **31** (1957), 195–218; MR **19** 1069] in a given complex. Using the cochains of the complex and some choice of equivariant chain mappings and equivariant homotopies, an explicit construction is given of cocycles mod 2 or mod 4, which are representatives of operations $\Phi_{2s}{}^i$ and $\Theta_{2s}{}^{2i+1}$, defined on cohomology classes belonging to the kernel of certain squares. A direct computation, using equivariant homotopies, shows that their cohomology class is not affected by different choices of the mappings or of the homotopies or of the complex W, nor by the choice of different representative cocycles in the cohomology classes of the complex; as the construction uses cochains which are defined only modulo arbitrary cocycles, $\Phi_{2s}{}^i$ and $\Theta_{2s}{}^{2i+1}$ map into factor-groups of the cohomology group (as can be expected for secondary cohomology operations). An explicit description is given of these factor-groups, as well as of the kernels on which the operations are defined.

The secondary operations Φ and Θ commute with the

homomorphisms induced by mappings, as is shown by a computation (in terms of cochains). They are not always additive; their action on a sum is computed. They do not always commute with the coboundary operator in relative cohomology; their action on this operator is computed. The corresponding functional operations are described.

Particular cases of these secondary operations have been considered before, by Adem [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 720–726; MR **14**, 306] for $\Phi_2{}^3$ and by N. Shimada and H. Uehara [Mem. Fac. Sci. Kyûshû Univ. Ser. A. **9** (1955), 37–46; MR **16**, 1142] for $\Theta_2{}^3$. No indication is given here about their application to topological problems (essential maps or third obstruction for the extension of a map). *G. Hirsch* (Brussels)

2232:

Copeland, Arthur H., Jr. Binary operations on sets of mapping classes. Michigan Math. J. **6** (1959), 7–23.

The author is mainly concerned to discuss two ways of putting a binary operation on the set of homotopy classes Map $[X, x_0; Y, y_0]$. If the binary operation is to be natural in X for fixed Y, it is equivalent to giving a class in Map $[Y^2, y_0{}^2; Y, y_0]$, that is, an H-space structure on Y. If the operation is to be natural in Y for fixed X, it is equivalent to giving a class in Map $[X, x_0; X \vee X, x_0{}^2]$, that is, a co-$H$-space structure on X. [Cf. P. J. Hilton, Fund. Math. **43** (1956), 358–386; MR **18**, 814; esp. § 3.] The author develops exact sequences corresponding to pairs and fiberings, and so forth.

By way of applications, he obtains, under slightly weaker hypotheses, the theorem that a fibre contractible in its space is an H-space [E. H. Spanier and J. H. C. Whitehead, Comment. Math. Helv. **29** (1955), 1–8; MR **16**, 610]. He also gives results (essentially known) on the classification of maps.

J. F. Adams (Cambridge, England)

2233:

★Whitehead, J. H. C. Duality between CW-lattices. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 248–258. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

This article, describing joint work of E. H. Spanier and the author, concerns the extension, from the suspension category of finite polyhedra to the suspension category of finite polyhedral lattices, of the duality theory developed by the authors. A duality between join-homomorphisms of complete lattices serves as the algebraic language for this relativization.

The notion of external n-duality between such lattices is introduced and investigated; this notion is exemplified by the following situation. Let $A_0 \subset A \subset X$ be CW-complexes, A_0 contractible, and X^*, A^*, $A_0{}^*$ be weak $(n-1)$-duals [Spanier and Whitehead, Mathematika **2** (1955), 56–80; MR **17**, 653] of X, A, A_0 such that the inclusion S-maps $X^* \to A^* \to A_0{}^*$ are dual to the inclusion S-maps $X \leftarrow A \leftarrow A_0$. Then $A_0{}^*/X^*$, $A_0{}^*/A^*$ are weakly n-dual to X, A, and $A_0{}^*/A^*$ may be identified with X'/A', $X' = A_0{}^*/X^*$, $A' = A^*/X^*$. Thus, the inclusion S-map $A \to X$ is dual to the pinching S-map $X' \to X'/A'$. This is external duality between the two lattices (X, A, A_0), (A_0', A', X'), where $A_0' = X^*/X^*$.

The main results are: (1) the existence of such duals, (2) a duality between corresponding exact couples, in particular, an isomorphism between the S-homotopy exact couple and the S-cohomotopy exact couple of duals, (3) the equivalence of extension and compression problems under this duality.

In the meantime, full details and proofs have appeared in another paper of the same authors [Ann. of Math (2) **67** (1958), 203–238]. *J.-P. Meyer* (Baltimore, Md.)

2234:

★Spanier, E. H. Duality and the suspension category. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 259–272. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

"This report is a summary of some of the joint work of J. H. C. Whitehead and the author on duality in homotopy theory." [See review above; also *Algebraic geometry and topology*, pp. 330–360, Princeton Univ. Press, 1957; MR **18**, 919.]

The author restricts himself to the absolute case and gives a sketch of the theory developed in the above-mentioned papers. In addition, he considers the following situation. Let X be a finite CW-complex with a single 0-cell, used as base-point; dim X = maximum dimension of positive-dimensional cells of X, codim X = minimum dimension of positive-dimensional cells of X, stem (X, Y) = dim X − codim Y, and $\{X, Y\}$ the group of S-homotopy classes of maps $X \to Y$. Let $\mathscr{C}$ denote a class of abelian groups closed under subgroups, quotient groups, extensions and isomorphisms [Serre, Ann. of Math. (2) **58** (1953), 258–294; MR **15**, 548] and H the homology (with coefficients in the abelian group G) functor from the suspension category of finite CW-complexes to the category of graded abelian groups. Assume G and $\mathscr{C}$ so related that for any $A \subset X$, and any Y, the sequences

$$\cdots \to \mathrm{Hom}\,(H(X/A), H(Y)) \to \mathrm{Hom}\,(H(X), H(Y)) \to \mathrm{Hom}\,(H(A), H(Y)) \to \cdots,$$

$$\cdots \to \mathrm{Hom}\,(H(Y), H(A)) \to \mathrm{Hom}\,(H(Y), H(X)) \to \mathrm{Hom}\,(H(Y), H(X/A)) \to \cdots$$

are $\mathscr{C}$-exact, and $H: \{X, Y\} \to \mathrm{Hom}\,(H(X), H(Y))$ is a $\mathscr{C}$-isomorphism for X, Y spheres of stem $\leq k$. Then H is a $\mathscr{C}$-isomorphism for arbitrary finite complexes X, Y, of stem $\leq k$. Special cases of this theorem give a theorem of Thom, and the fact that $\{X, Y\}$ is finitely-generated, for X, Y finite complexes.

An application of the duality theory to imbedding problems is indicated. This idea has been further exploited by F. P. Peterson [Bol. Soc. Mat. Mexicana (2) **2** (1957), 9–15; MR **19**, 440]. *J.-P. Meyer* (Baltimore, Md.)

2235:

Hu, Sze-Tsen. On fibre spaces with cross-sections. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 170–176.

Assuming the existence of a cross-section over the base space B of a fibre space E, one identifies B with this cross-section, and the cross-section $q: B \to E$ becomes the inclusion map. For a subspace A of B, let $D = p^{-1}(A)$ (where $p: E \to B$ is the projection); let A_0 be a subspace of A and $a_0 \in A_0$ the base point for all homotopy groups.

The author constructs a sequence

$$\cdots \leftarrow \pi_n(E, A_0) \xleftarrow{\varphi} \pi_n(B, A_0) \times \pi_n(D, A_0) \xleftarrow{\psi} \pi_n(A, A_0) \xleftarrow{\Delta} \pi_{n+1}(E, A_0) \leftarrow \cdots$$

which is shown to be exact, ending in $\pi_2(E, A_0)$ in the relative case [or in $\pi_1(E)$ in the absolute case (where one takes $A_0 = a_0$)]; the first φ is not always a homomorphism, and the first three groups might not be abelian. Using the inclusions $A \subset B \subset E$ and $A \subset D \subset E$, a diagram similar to the Mayer-Vietoris sequence [S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N.J., 1952; MR **14**, 398; p. 39] is used for the definition of the mappings φ, ψ and Δ. The exact homotopy sequences of $F \to E \to B$ and of the triplet E, B, a_0 show that $\pi_n(E, D) \simeq \pi_n(B, A)$ and $\pi_n(E, B) \simeq \pi_n(D, A)$ $(n \geq 1)$.

φ is a homomorphism for $n = 2$ if $\pi_2(E, A_0)$ (in the relative case) [or for $n = 1$ if $\pi_1(E)$ (in the absolute case)] is abelian; it maps onto $\pi_2(E, A_0)$ for $n = 2$ if $\pi_1(A, A_0)$ consists of a single element (in the relative case) [or it maps onto $\pi_1(E)$ for $n = 1$ if $A = a_0$ (in the absolute case)].

G. Hirsch (Brussels)

2236:

van Est, W. T. A generalization of the Cartan-Leray spectral sequence. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 399–413.

Let $\{M, G, p, B\}$ be a principal fibre bundle. When G is discrete, Cartan and Leray [Topologie algébrique, pp. 83–85, Centre de la Recherche Scientifique, Paris, 1949; MR **11**, 677] defined a spectral sequence connecting the topological cohomology of B and the group cohomology of G. This is generalized by the author so as to include the case in which G is a Lie group and the bundle a differentiable bundle. In the Lie group case, the group cohomology of G is that based on the differentiable cochains. By means of this generalization, the transfer introduced by Eckmann and the spectral sequence of Hochschild-Serre for group extensions carry over from the discrete to the Lie group case. As application to special problems, the author proves the following. (i) The Betti numbers of a discrete subgroup of a Lie group with compact coset space are bounded below by the algebraic Betti numbers of the total group (Betti numbers are used in the sense of the group cohomology with real coefficients and trivial action). (ii) Let M be a nilpotent Lie group such that the component group is finitely generated, G a discrete subgroup of M with M/G compact, and π a nilpotent representation of M over a real vector space V. Then the restriction map: $H^*(M, V, \pi) \to H^*(G, V, \pi)$ is an isomorphism onto.

H. C. Wang (Evanston, Ill.)

2237a:

Fadell, Edward; and Hurewicz, Witold. On the structure of higher differential operators in spectral sequences. Ann. of Math. (2) **68** (1958), 314–347.

2237b:

★Hurewicz, Witold; and Fadell, Edward. On the structure of higher terms of the spectral sequence of a fibre space. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 1–15. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

Both papers give a detailed proof of a theorem which generalizes a theorem of Leray-Serre. If (X, B, p) is a fibre space in the sense of Hurewicz [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961; MR **17**, 519] with fibre F, and if B is $(r-1)$-connected $(r > 1)$, then in the associated spectral sequence (E^i, d^i), E^i is isomorphic with $H_*(B, H_*(F; G))$ for $2 \leq i \leq r-1$; d^r is given by the cap product with the characteristic cohomology class of B; the product is defined in terms of a pairing of $\pi_r(B)$ and $H_*(F; G)$. This pairing is defined as follows: let Ω_p consist of all pairs (w, x), where $x \in X$ and w is a path of B, starting from $p(x)$; the fibre space admits a lifting function λ, mapping Ω_p into the space of paths of X (where $\lambda(w, x)$ is a path starting from x and such that $p\lambda(w, x) = w$). Let $\bar{B}$ denote the space of paths of B, starting from a base point b; denote by Ω the space of loops in B with base point b, and let $F = p^{-1}(b)$; λ induces $\bar{\lambda}: \bar{B} \times F \to X$ and $\Omega \times F \to F$, by mapping (w, f) onto the end point of the lifted path $\lambda(w, f)$ for each $f \in F \subset X$. From $\bar{\lambda}_*: H_*(\Omega \times F) \to H_*(F)$ results first a pairing of $H_{r-1}(\Omega)$ and $H_n(F; G)$ to $H_{n+r-1}(F; G)$; one identifies $\pi_r(B)$ with $\pi_{r-1}(\Omega)$, which is mapped into $H_{r-1}(\Omega)$.

The proof results in the first paper from a comparison of the filtered group of special singular prismatic chains of X and of the group $C_*(B) \otimes C_*(F) \otimes G$, filtered by the obvious filtration given by the dimension of the first factor; because B is $(r-1)$-connected, their spectral sequences are isomorphic for $1 \leq i \leq r$; the first group yields the homology of X and its spectral sequence is isomorphic with the spectral sequence based on singular cubical chains. d^r is computed with the auxiliary fibre space $(\bar{B} \times F, B, p\bar{\lambda})$.

The corresponding result for cohomology is also given. An example (with $S^4 \times S^4$ as base space and S^7 as fibre) shows that knowledge of the pairing $H_*(\Omega) \otimes H_*(F) \to H_*(F)$ is not sufficient to determine the higher differential operators d^i (for $i > r$).

In the second paper, special singular cubical chains are used to prove the main theorem.

The theorem was announced in earlier papers [ibid. **41** (1955), 961–964; **43** (1957), 241–245; MR **17**, 520; **18**, 752].

G. Hirsch (Brussels)

2238:

Rohlin, V. A. Internal homologies. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 876–879. (Russian)

The author characterises the image of the oriented cobordism ring $\mathcal{O}$ in the unoriented cobordism ring $\mathcal{N}$ under the natural map h. Let $\{P(r)\}$, for r not of the form $2^s - 1$, be a system of generators of $\mathcal{N}$. For even r, $P(r)$ is real projective space $\mathrm{PR}(r)$, for odd r it can be taken orientable. It is first shown, by considering Stiefel-Whitney numbers, that $\mathrm{PR}(n) \times \mathrm{PR}(n) = \mathrm{PC}(n)$ in $\mathcal{N}$ (here $\mathrm{PC}(n)$ = complex projective space). A product of generators $P(r)$ in which each even-dimensional factor appears an even number of times belongs to $h(\mathcal{O})$ and is called of first kind. The remaining products are called of second kind; products without repeated factors are called special. A Stiefel-Whitney number that involves the Stiefel-Whitney class w_1 is called a w_1-weight. Five sets of manifolds are introduced: $M \in K_1$, if the $\mathcal{N}$-class of M belongs to $h(\mathcal{O})$; $M \in K_2$, if every w_1-weight is 0; $M \in K_3$,

if the $\mathcal{N}$-class of M is generated by products of the first kind; $M_k \in K_4$, if the $\mathcal{N}$-classes of A_{k-1} and B_{k-2} vanish [A_{k-1}, B_{k-2} are submanifolds determined using w_1; cf. V. A. Rohlin, same Dokl. **89** (1953), 789–792; MR **15**, 53]; $M \in K_5$ if the $\mathcal{O}$-class of A_{k-1} and the $\mathcal{N}$-class of B_{k-2} are zero.

Theorem: $K_1 = K_2 = K_3 = K_4 = K_5$.

Some of the inclusion relations are obvious. For the others short proofs are indicated. Corollaries: (1) In $\mathcal{N}$ the product of a nonorientable element by a nontrivial orientable element is nonorientable; (2) if M is 0 in $\mathcal{N}$, and $2M$ is 0 in $\mathcal{O}$, then M is 0 in $\mathcal{O}$; (3) $\mathcal{O}$ contains no elements of order 4. {Reviewer's remark: These results are similar to, but apparently not in complete agreement with those of C. T. C. Wall [Bull. Amer. Math. Soc. **65** (1959), 329–331].} *H. Samelson* (Ann Arbor, Mich.)

2239:

Shimada, Nobuo. On the differentiable structure of spherical surfaces. Sûgaku **9** (1957/58), 85–95. (Japanese)

As is well known J. Milnor proved that the 7-sphere admits at least four different C^∞-structures [Ann. of Math. (2) **64** (1956), 399–405; MR **18**, 498].

The author proves that the 15-sphere admits at least 64 different C^∞-structures and states applications to the determination of the homotopy types of a certain sphere bundle. *K. Yano* (Tokyo)

2240:

★Milnor, John. On simply connected 4-manifolds. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 122–128. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

The author notes that as a consequence of a theorem of J. H. C. Whitehead [Comment Math. Helv. **22** (1949), 48–92; MR **10**, 559] two oriented simply connected 4-manifolds are of the same oriented homotopy type if and only if their quadratic forms are equivalent. Here the quadratic form refers to the pairing of $H^2(M, Z)$ with itself to the integers, assigning to a pair of elements the value of their cup product on M. Equivalence means an isomorphism of the second cohomology groups that preserves the pairing.

For abstract quadratic forms over the integers one has the concepts of genus (equivalence over all p-adic integers and over the reals), rank, index (or signature), type (I, if some diagonal entry of the matrix is odd, II otherwise). The author proves that rank, index and type determine the genus of a form of determinant ± 1, and gives the relations holding between the three characteristics. Every such form is equivalent over the integers either to a diagonal matrix with $+1$'s and -1's as diagonal entries, or to plus or minus a sum of U's and V's where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and V is a certain explicit 8×8 matrix (of type II and index 8). Further, two indefinite forms of determinant ± 1 are equivalent exactly if they have the same genus; for definite forms this holds for rank ≤ 8.

These theorems are then applied to the manifolds above. Sample corollary: A manifold of type II and index 0 has the homotopy type of a sum of copies of $S^2 \times S^2$. An algebraic manifold of second Betti number 22, index -16, type II is described. It is unknown whether every quadratic form of determinant ± 1 is represented by some manifold. *H. Samelson* (Ann Arbor, Mich.)

2241:

Whitehead, J. H. C. On 2-spheres in 3-manifolds. Bull. Amer. Math. Soc. **64** (1958), 161–166.

The sphere theorem of Papakyriakopoulos [Ann. of Math. (2) **66** (1957), 1–26; MR **19**, 761] is generalized to the following: Let M be a connected, orientable, triangulated 3-manifold in a topological space X. If the image of the injection $\pi_2(M) \to \pi_2(X)$ is non-trivial then M contains a non-singular, polyhedral 2-sphere that is not contractible in X. When $X = M$ this reduces to the sphere theorem freed of a certain restrictive hypothesis required in the original proof. Several corollaries are deduced: (1) If the interior of M is open in X (and X is Hausdorff) then the kernel of the injection $\pi_1(M) \to \pi_1(X)$ is locally infinite; this is even true if $\pi_1(M)$ is replaced by $\pi_1(A)$ for A any compact connected ANR in M. (2) If M is compact then there is a finite disjoint set of non-singular, polyhedral 2-spheres whose transforms by elements of $\pi_1(M)$ generate $\pi_2(M)$. *R. H. Fox* (Princeton, N.J.)

2242:

Shapiro, Arnold; and Whitehead, J. H. C. A proof and extension of Dehn's lemma. Bull. Amer. Math. Soc. **64** (1958), 174–178.

Let $C_1, \cdots, C_r$ be a disjoint set of simple closed polygons in a connected triangulated 3-manifold M such that the covering transformations that they induce in the universal covering manifold $\tilde{M}$ are all orientation-preserving (as is always the case if M is orientable). Suppose that in M there is a polyhedral surface of genus zero bounded by $C_1 + \cdots + C_r$ with only interior singularities. Then there must exist a non-singular polyhedral surface of genus zero whose boundary is a subset $C_{i_1} + \cdots + C_{i_q}$ of $C_1 + \cdots + C_r$ $(0 < q < r)$. For $r = 1$ this is just Dehn's lemma, and in this case the author's proof is a simplified version of the Papakyriakopoulos proof [op. cit., review above]. (The simplification is essentially the use of 2-sheeted rather than universal coverings.)

R. H. Fox (Princeton, N.J.)

2243:

Livesay, G. R. On maps of the three-sphere into the plane. Michigan Math. J. **4** (1957), 157–159.

This note contains the proof of the following theorem: Let $f: S^3 \to E^2$ be a continuous map of the 3-sphere into the Euclidean plane, and let p_1, p_2, and p_3 be points of S^3 which are vertices of an equilateral triangle. Then there exists a rotation r of S^3 such that $f(rp_1) = f(rp_2) = f(rp_3)$. This theorem was motivated by a conjecture of B. Knaster [see Colloq. Math. **1** (1948), 30–31].

The author has pointed out in a letter to the reviewer that the same method may be used to prove this theorem for maps of an $(n+1)$-sphere into Euclidean n-space. The theorem for this more general case is actually stated and proved in a paper by C. T. Yang which appeared about the same time as the note under review [see Amer. J. Math. **79** (1957), 725–732].

W. S. Massey (Providence, R.I.)

2244:

Kosiński, A. On involutions and families of compacta. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1055–1059, LXXXVII-LXXXVIII. (Russian summary)

Let M be a compact metric space and let φ be an involution on M, i.e., a continuous mapping $\varphi: M \to M$ such that $\varphi\varphi(x) = x$. Let T be the set of fixed points of M under φ. The author proves that if $\dim M \leq n$ and $\dim T \leq n-1$, then there are two compact subsets F_1 and F_2 of M such that $F_1 \cup F_2 = M$, $\varphi(F_1) = F_2$, and $\dim (F_1 \cap F_2) \leq n-1$. If φ is an involution on M, denote by M^* the space obtained from M by identifying pairs $(x, \varphi(x))$ and let k be the natural mapping of M into M^* defined by $k(x) = (x, \varphi(x))$. A consequence of the above is Th. 1: If M, φ, T are as above and the automorphism $\varphi_*: H_n(M) \to H_n(M)$ ($H_n(X)$ is the nth Čech homology group of X over the integers mod 2) is the identity, the homomorphism $k_*: H_n(M) \to H_n(M^*)$ induced by k is trivial.

Th. 1 is used to generalize theorems due to Borsuk and to Borsuk and the author. If X is compact metric and Φ is an upper semi-continuous mapping of X into the space of all nonempty compact subsets of a metric space Y, call the triple $F = \{X, Y, \Phi\}$ a family. The field of F is the set $\bigcup_{x \in X} \Phi(x) \subset Y$. F is said to be acyclic if all the sets $\Phi(x)$ are connected and $H_k(\Phi(x)) = 0$ for $k \geq 2$ and all $x \in X$. A family $F_1 = \{X, Y, \Phi_1\}$ is a prolongation of a family $F_2 = \{X, Y, \Phi_2\}$ if $\Phi_1(x) \supset \Phi_2(x)$ for all $x \in X$. If $\varphi: M \to M$ is an involution, then $F = \{M^*, E, k^{-1}\}$ is a family for every $E \supset M$ (compact) and is called a family induced by φ. Th. 2: Assume that M is a compact $(n-1)$-dimensional subset of E^n, $\varphi: M \to M$ is an involution, $F_1 = \{M^*, E^n, k^{-1}\}$ is the family induced by φ, $H_{n-1}(M)$ is the group of integers mod 2, and that the set of fixed points of φ is of dimension $\leq n-2$. Then the field of each acyclic prolongation $F = \{M^*, E^n, \Phi\}$ of F_1 contains all the bounded components of $E^n - M$. Th. 3: Let Q be an n-cell and S its boundary. Suppose that $\varphi: S \to S$ is an involution without a fixed point and that $F_1 = \{S^*, Q, k^{-1}\}$ is the family induced by φ. Then if $F = \{S^*, Q, \Phi\}$ is any acyclic prolongation of F_1, there exists a point in Q which is contained in three distinct elements of F, i.e., in three distinct sets $\Phi(x)$, $x \in S^*$.

Th. 2 is a generalization of a theorem of Borsuk [Fund. Math. **42** (1955), 240–258, p. 256; MR **17**, 881] and a theorem due to Borsuk and the author [same Bull. **3** (1955), 293–296, Th. 3c; MR **17**, 654]. Th. 3 generalizes another one of the author [Fund. Math. **46** (1958), 47–59].

H. Komm (Troy, N.Y.)

2245:

Wu, Wen-tsün. On the dimension of a normal space with countable basis. Sci. Record (N.S.) **2** (1958), 65–69.

In a preceding paper of the author [Sci. Record N.S. **1** (1957), 377–380; MR **20** #6097] the notions of $\Phi_{(p)}$-classes and $\Phi_{(p)}$-indices for a normal space with countable basis are introduced. He has also shown that these classes serve to characterize the dimension of a finite polyhedron P as the smallest integer $n \geq 0$ such that

$$\Phi_{(p)}{}^{(n+1)(p-1)-1}(P) = 0$$

for any prime p sufficiently large. In other words, the dimension of a finite polyhedron P is equal to $[I_p(P)/(p-1)]$ for p sufficiently large, where $[x]$ denotes the greatest integer $\leq x$. In the present note, the author proves that the same is true for the dimension of any normal space with countable basis if the notions of $\Phi_{(p)}$-classes and $\Phi_{(p)}$-indices are slightly modified.

Sze-tsen Hu (Palo Alto, Calif.)

2246:

Lashof, R.; and Smale, S. On the immersion of manifolds in euclidean space. Ann. of Math. (2) **68** (1958), 562–583.

Für eine Immersion der k-dimensionalen, zusammenhängenden, kompakten, orientierten, differenzierbaren Mannigfaltigkeit M^k in den Euklidischen Raum E^{k+l} ist die Euler- (Stiefel-Whitney-)sche Klasse $\overline{W}_l \in H^l(M, Z)$ des Normalbündels definiert, deren Restriktion mod 2 bekanntlich nicht von der Immersion abhängt. Für $l = k$ ist $\overline{W}_k \equiv 0 \bmod 2$. Ferner ist die Eulersche Klasse $W_k \in H^k(M, Z)$ des Tangentialbündels von M definiert. Für eine Immersion $f: M^k \to E^{k+l}$ lässt sich der Normalgrad von f einführen, als der Grad der Abbildung, welche jeden Vektor des Normalbündels durch Parallelverschiebung in den Ursprung des E^{k+l} überführt. Für $l = k$ führt man entsprechend den Tangentialgrad ein. Verff. zeigen: (a) Für eine Immersion $M^k \to E^{2k}$ ist der Tangentialgrad gleich $\pm \overline{W}_k \cdot M$ (Wert von $\overline{W}_k$ auf dem Grundzyklus der orientierten M). (b) Für eine Immersion $M^k \to E^{k+l}$ mit $l > 1$ ist der Normalgrad gleich $\pm W_k \cdot M$ (bei geeigneter Wahl der Orientierungen also gleich der Eulerschen Charakteristik von M). Die Aussage (b) ist bekannt [Chern, Enseignement Math. **40** (1955), 26–46; MR **16**, 856], wird aber von den Verff. in verallgemeinerter Form bewiesen: Verff. definieren für eine Immersion $f: M^k \to M'^{k+l}$, wobei M' nicht kompakt zu sein braucht ($l > 1$), einen Normalgrad, welcher eine ganze Zahl modulo $W'_{k+l} \cdot M'$ ist. Hierbei bezeichnet W'_{k+l} die tangentielle Eulersche Klasse von M', und $W'_{k+l} \cdot M'$ ist ihr Wert auf dem Grundzyklus von M'. Dieser Wert wird gleich 0 gesetzt, wenn M' nicht kompakt ist. Die Aussage (b) bleibt bestehen mod $W'_{k+l} \cdot M'$.

Die Verff. beweisen, dass der Tangentialgrad einer vollständig regulären Immersion $M^k \to E^{2k}$ (k gerade) gleich dem 2-fachen der von Whitney eingeführten Selbstschnittzahl ist [Ann. of Math. (2) **45** (1944), 220–246; MR **5**, 273]. Aus Resultaten von Whitney folgt, dass jede gerade ganze Zahl als $\overline{W}_k \cdot M$ (M fest vorgegeben von der geraden Dimension k), einer Immersion $M^k \to E^{2k}$ auftritt. Die Beweise der referierten Resultate erfolgen mit Hilfe Gysinschen Sequenzen. Die Gysinsche Sequenz der Whitneyschen Summe zweier Sphärenbündel wird zu den Gysinschen Sequenzen der Summanden in Beziehung gesetzt.

Die Verff. untersuchen den ganzzahligen Cohomologiering der Grassmannschen Mannigfaltigkeit $G(k, l)$ der orientierten k-Ebenen durch den Ursprung des E^{k+l} und die Homomorphismen $i^*: H^*(G(k, \infty), Z) \to H^*(G(k, l), Z)$. Sie wenden ihre Ergebnisse auf die Tangentialabbildung $t: M^k \to G(k, l)$ an, die einer Immersion $M^k \to E^{k+l}$ zugeordnet ist. Für $l \leq k$ erhalten sie für ganzzahlige Cohomologie: "Theorem. Let f and g be immersions of M^k in E^{k+l} with $l > 1$ if k is odd, and with the same normal Euler class $\overline{W}_l$. Then the induced homomorphisms of the tangential maps $t^*: H^*(G(k, l)) \to H^*(M)$ of f and g are the same."

Neben anderen Resultaten geben Verff. einen neuen Beweis für einen Satz von Kervaire [Math. Ann. **131** (1956), 219–252; MR **19**, 160].

{Hinsichtlich des Korollars 5.2 möchte Ref. bemerken,

dass man mit anderen Methoden folgendes beweisen kann: Die $4m$-dimensionale, kompakte, orientierte, differenzierbare Mannigfaltigkeit M sei in E^{4m+4} eingebettet. Dann ist die Pontrjaginsche Klasse der Dimension $4m$ von M durch $(2m+1)!/2$ teilbar und für gerades m sogar durch $(2m+1)!$ teilbar, und die Eulersche Charakteristik von M ist gerade für $m \geqq 2$.} *F. Hirzebruch* (Bonn)

2247:

Kinoshita, Shin'ichi. Notes on knots and periodic transformations. Proc. Japan Acad. **33** (1957), 358–362.

The results of this note are all contained in the paper reviewed below. Moreover there is an error in § 1, in that $v(i)$ is not well-defined. However, all the results are correct, and the difficulty about $v(i)$ disappears if it is assumed from the beginning that M is a Poincaré space. *R. H. Fox* (Princeton, N.J.)

2248:

Kinoshita, Shin'ichi. On knots and periodic transformations. Osaka Math. J. **10** (1958), 43–52.

In this paper the note reviewed above is reworked and amplified. Let T be a sense preserving semilinear transformation of S^3 upon itself of period $g>1$ whose set of fixed points is a simple closed curve F. The orbit space is then a simply connected 3-dimensional manifold M and S^3 is a p-fold cyclic covering of M branched along F. Denote the Alexander polynomial of $\pi(S^3-F)$ by $\Delta_g(x)$. The principal results are the following. Theorem 5: If $g=2$ then (i) the determinant $|\Delta_2(-1)|$ of F must be equal to the square of an odd number, (ii) the degree of $\Delta_2(x)$ must be $\geqq 4$, (iii) F cannot be any of the prime knots of fewer than 10 crossings except possibly 8_9 or 8_{20}. Theorem 6: If $g=3$ then (i) the degree of $\Delta_3(x)$ must be $\geqq 4$, (ii) F cannot be any of the prime knots of fewer than 10 crossings except possibly 5_1, 7_1, 8_{10} or 9_{47}. *R. H. Fox* (Princeton, N.J.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 1954, 2021.

2249:

Wunderlich, W. Äquidistante Kurvenpaare in normalen Ebenen. Arch. Math. **10** (1959), 64–70.

Two curves k_1 and k_2 in euclidean 3-space form an equidistant pair if they possess a family of common normals. The author studies those pairs k_1, k_2 for which k_1 and k_2 lie in the (x, y)- and (x, z)-plane respectively. They depend on one arbitrary function of x. Many special cases are discussed. *P. Scherk* (Boulder, Colo.)

2250:

Sispánov, Sergio. Darboux curves on surfaces of rotation. Rev. Mat. Cuyana **1** (1955), 169–180. Also in Acta Cuyana Ingen. **1**, no. 9, 17–29 (1957). (Spanish)

The author deduces the differential equations for the Darboux curves (i.e., curves whose osculating sphere is everywhere tangent to the tangent plane of the surface) on surfaces of revolution. The equations are integrable by quadratures. Applications: (a) If the Darboux curves are helices, the surface is a quadric of revolution (Blaschke); (b) if they are loxodromic curves, the meridian curves of the surface are circles or straight lines [see L. A. Santaló, Publ. Inst. Mat. Univ. Nac. Litoral **5** (1945), 255–267; MR **7**, 172]. *L. A. Santaló* (Buenos Aires)

2251:

Drăgilă, Pavel. Correspondance entre deux surfaces par trois couples de tangentes parallèles. Proc. Amer. Math. Soc. **9** (1958), 851–854.

The author continues his studies on the correspondence of two surfaces already started in a previous paper [same Proc. **9** (1958), 189–200; MR **20** #2339] in which he proved that there exist couples of surfaces which correspond by parallelism of tangent planes and by parallelism of a single system of tangents. He actually determined the couples of such surfaces by integration of the system $\bar{r}_u = r_u - r_v$, $\bar{r}_v = kr_u$ for $k > \frac{1}{4}$. In this note the case $k < \frac{1}{4}$ is investigated, and in that case the existence of corresponding surfaces with three couples of corresponding tangents is established. *E. T. Davies* (Southampton)

2252:

Drăgilă, Pavel. Correspondance entre deux surfaces par quatre couples de tangentes parallèles. Proc. Amer. Math. Soc. **9** (1958), 855–859.

The author considers surfaces whose asymptotic directions are everywhere orthogonal. Such surfaces exist and are minimal surfaces. Upon these surfaces asymptotic directions bisect the angles formed by the principal directions of the surfaces. Let S be a minimal surface referred to the lines of curvature u and v, and let us find the second surface $\bar{S}$ obtained by the transformation $\bar{r}_u = \lambda r_v$, $\bar{r}_v = \mu r_u$. $\bar{S}$ will thus be referred to the asymptotic lines u and v which are orthogonal, proving that $\bar{S}$ is also a minimal surface. The asymptotic directions on the first surface S and the principal directions upon the second surface $\bar{S}$ must also be parallel, since they are the bisectors of two angles with parallel sides. In this way the present paper continues the author's study of correspondences between surfaces with couples of parallel tangents. It also reveals an interesting transformation of minimal surfaces. The paper deals later with the problem of determining minimal surfaces which are referred to their asymptotic lines. *E. T. Davies* (Southampton)

2253:

Drăgilă, Pavel. Sur une classe de surfaces. Boll. Un. Mat. Ital. (3) **13** (1958), 465–469. (Italian and English summaries)

A study is made of the set of surfaces in ordinary space which are defined by the system of three partial differential equations

$$r_{uu} = ar_u + br_v + cr,$$

with corresponding expressions for r_{uv} and r_{vv}. These surfaces are equivalent from the affine point of view. The author examines couples of surfaces S and $\bar{S}$ of the set with the property that the direction of the tangent to a coordinate line upon S shall be parallel to the direction of the tangent at the corresponding point on $\bar{S}$. A special study is made of the case in which the directions are asymptotic. *E. T. Davies* (Southampton)

2254:

Laugwitz, Detlef. Über die Flächen mit mehreren Scharen von ebenen geodätischen Linien. Arch. Math. **8** (1957), 472–476.

Let K_n denote the class of those surfaces which possess n families of plane geodesics. The surfaces in K_4 lie on planes or spheres; a surface in K_3-K_4 lies on a hyperboloid of revolution of one sheet; a surface in K_2-K_4 lies on a general cylinder or on a ruled quadric or it is a (non-developable) ruled surface with a family of plane lines of curvature.

Suppose the surfaces $\mathfrak{x}(u, v)$ and $\mathfrak{y}(u, v)$ have parallel tangent planes at corresponding points. A curve on $\mathfrak{x}(u, v)$ is called a relative geodesic if its osculating plane is always parallel to the "relative normal" $\mathfrak{y}(u, v)$ [W. Süss, Japan. J. Math. **4** (1927), 57–75]. If $\mathfrak{x}(u, v)$ has at least three families of plane relative geodesics none of which are straight lines, then $\mathfrak{x}(u, v)$ is a "relative sphere", i.e., $\mathfrak{x}(u, v)$ and $\mathfrak{y}(u, v)$ are homothetic.

P. Scherk (Boulder, Colo.)

2255:

Jha, P. On the characteristic equation of a rectilinear congruence. Math. Student **26** (1958), 9–16.

2256:

Ribeiro de Albuquerque, J. The developable surfaces and the envelopes of the faces of the moving trihedral. I, II. Gaz. Mat., Lisboa, **18** (1957), 17–23; **19** (1958), 12–18. (Portuguese)

Il s'agit ici, essentiellement, d'un article de caractère didactique. L'A. expose, d'une façon claire et synthétique, plusieurs aspects de la géométrie des lignes, compris dans le programme des cours classiques d'analyse infinitésimale.

J. Sebastião e Silva (Lisbon)

2257:

Poliščuk, E. M. Metrical properties of isogenous functionals and vector fields. Vestnik Leningrad. Univ. **12** (1957), no. 13, 27–49. (Russian. English summary)

This paper begins with discussion of derivatives which are functions of curves [in the sense of Volterra, Acta Math. **12** (1899), 233–286]. The resulting conditions applied to vector-valued functions lead the author to study complex vector fields $A=A^1+iA^2$, where each A^j is a real vector field in E_3, with $\operatorname{div} A^j=0$; for L a smooth closed curve in E_3 he sets $F_A=F_A(L)=\iint_\Sigma A\cdot nd\sigma$, where Σ is a smooth oriented surface spanning L, n is its unit normal vector, and integration is with respect to surface area on Σ.

If f is a scalar field in E_3, the fields $F=F_A$ and $\Phi=F_{fA}$ are called isogenous and the field f is called the derivative $d\Phi/dF$; if $B=fA=B^1+iB^2$, the fields B are called conjugate. A Riemannian space V_3 is defined by calculating the components of a metric tensor by proper combinations of the components of A. Then theorem I says that if $d\Phi/dF=\rho e^{i\varphi}$, then at each point of V_3 the vectors B^1 and B^2 are orthogonal and have the length $\rho\sqrt{3}$. Theorem II asserts that the Riemannian spaces defined by A and by B are conformal. The last section discusses harmonic fields.

Some analogous results are obtained for vector fields $A^1+\epsilon A^2$, where $\epsilon^2=1$.

M. M. Day (Urbana, Ill.)

2258:

★Raschewski, P. K. Riemannsche Geometrie und Tensoranalysis. Hochschulbücher für Mathematik, Bd. 42. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 606 pp. DM 42.00.

A translation by W. Richter from the Russian [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 1051].

2259:

Peña López, Roberto. A property of conservative fields. Rev. Soc. Cubana Ci. Fís. Mat. **4** (1957/58), 64–65. (Spanish)

Se per un punto mobile $P=P(t)$ riesce $\dot{P}=-\operatorname{grad} W(P)$ è allora $\ddot{P}=\operatorname{grad}\{\frac{1}{2}|\operatorname{grad} W|^2\}$.

G. Fichera (Madison, Wis.)

2260:

Strubecker, Karl. Zur Vektoranalysis auf Flächen. Arch. Math. **9** (1958), 123–127.

Let φ be a function on a surface $S\subset E_3$, P a point of S, B a neighborhood of P whose area is $O(B)$ and whose boundary is R, H the mean curvature of S, $\mathbf{N}$ the unit normal to S at P, and $\mathbf{n}$ the unit normal to S in B. Then it is proved that

$$\lim_{B\to P}\frac{\int_R \varphi d\mathbf{n}}{O(B)}=(\operatorname{grad}\varphi)_P+(2\varphi H)_P\mathbf{N},$$

where grad φ is interpreted as a vector in E_3.

C. B. Allendoerfer (Seattle, Wash.)

2261:

Singh, Har Dutt. An extension of a covariant differentiation process. II. Tensor (N.S.) **8** (1958), 90–94.

[For part I, see Tensor (N.S.) **7** (1957), 137–140; MR **20** #2738.]

We consider a general tensor whose components are not only functions of the coordinates x but also functions of m derivatives $x', x'', \cdots, x^{(m)}$ of the coordinates x with respect to the parameter. The covariant derivative of such a tensor was defined by H. V. Craig, M. M. Johnson and the author himself.

In the present paper, the author finds a formula defining the covariant derivative of a tensor whose components are of the form $T_\alpha^\beta(x, x', x'')$ and he compares his result with the former result in this special case.

K. Yano (Tokyo)

2262:

Barner, Martin. Über geschlossene Laplace-Ketten. Arch. Math. **9** (1958), 366–377.

The parameter curves $u=\text{const}$ and $v=\text{const}$ of a two-dimensional surface $\mathfrak{x}(u, v)$ in projective $R_\mathfrak{x}$ form a conjugate net if the points $\mathfrak{x}_{uv}, \mathfrak{x}_u, \mathfrak{x}_v, \mathfrak{x}$ are linearly dependent, say $\mathfrak{x}_{uv}+a\mathfrak{x}_u+b\mathfrak{x}_v+c\mathfrak{x}=0$. Their tangents then also envelop the surfaces $\mathfrak{x}_1=a\mathfrak{x}+\mathfrak{x}_v$ and $\mathfrak{x}_{-1}=b\mathfrak{x}+\mathfrak{x}_u$ whose parameter curves again form conjugate nets. Repeating this procedure as often as possible, we obtain a Laplace chain of conjugate nets. [Cf. G. Bol, *Projektive Differentialgeometrie*, vol. I, Vandenhoeck and Ruprecht, Göttingen, 1950; MR **11**, 539; pp. 116–125.]

If the surfaces $\cdots\mathfrak{x}_0, \mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{x}_{n+1}, \cdots$ form a Laplace chain such that $\mathfrak{x}_i=\mathfrak{x}_{i+n+1}$ for $i=0$ (and hence for all i), the chain is said to be closed and to have the period

$n+1$. Following Bol [Math. Z. **54** (1951), 141–159; MR **13**, 160] the author introduces $n+1$ differential forms π_i which are transformed into $\pi_i^* = \pi_i - d(\log \rho_i)$ if the ξ_i are replaced by $\rho_i \xi_i$ $(i = 0, 1, \cdots, n)$. If the form or vector g has the transformation law $g^* = \prod \rho^{c^i} g$, define the semi-invariant exterior derivative of g through

$$d^v g = dg + \sum_0^n c_i \pi_i \wedge g.$$

Here dg is the ordinary exterior derivative {= differential} and $\wedge$ indicates the alternating product. The π_i can be chosen such that (1) $d^v \xi_i = \alpha_i \xi_{i-1} + \beta_i \xi_{i+1}$. The α_i, β_i then satisfy the integrability conditions (2) $d^v \alpha_i = d^v \beta_i = d^v \pi_i + \alpha_i \wedge \beta_{i-1} + \beta_i \wedge \alpha_{i+1} = 0$. Conversely if the α_i and β_i satisfy (2), (1) has one and only one solution with given initial values. Hence the closed Laplace chains C_k^{n+1} of period $n+1$ in R_k depend on $2n$ arbitrary functions of one variable. To every C_k^{n+1} there exists a C_n^{n+1} with the same forms α_i, β_i, π_i; C_n^{n+1} is unique up to a projectivity and C_k^{n+1} is its projection.

The intersection of a C_n^{n+1} with an R_{n-1} is a C_{n-1}^{n+1}. Conversely any $C_{n-1}^{n+1} \in R_{n-1}$ which does not lie in a linear subspace is the intersection of R_{n-1} with a C_n^{n+1} which is again unique up to projectivities.—In the last section the C_{2m-1}^{2m+1} are studied with $\alpha_i \wedge \beta_{i-1} = \alpha_{i+m} \wedge \beta_{i+m}$ $(i = 0, 1, \cdots, 2m)$. *P. Scherk* (Toronto, Ont.)

2263:

Su, Buchin. Laplace sequences associated with a projective minimal surface. Sci. Record (N.S.) **2** (1958), 183–188.

Let

$$(J) \qquad \cdots, J_n, \cdots, J_2, J_1, J, J_{-1}, J_{-2}, \cdots, J_{-n}, \cdots,$$

$$(\bar{J}) \qquad \cdots, \bar{J}_n, \cdots, \bar{J}_2, \bar{J}_1, \bar{J}, \bar{J}_{-1}, \bar{J}_{-2}, \cdots, \bar{J}_{-n}, \cdots$$

be two associate Laplace sequences of a projective minimal surface S in a three-dimensional projective space, and let

$$(G) \qquad \cdots, A_n, \cdots, A_2, A_1, A, B, B_1, B_2, \cdots, B_n, \cdots$$

be the common conjugate sequence of (J) and $(\bar{J})$. It is proved that the joins $J_{n+2}J_{n+3}$ and $J_{-n+1}J_{-n+2}$ meet the joins $\bar{J}_{n-2}\bar{J}_{n-1}$ and $\bar{J}_{-n-2}\bar{J}_{-n-3}$ at A_n and B_n, respectively, such that the sequence (G) coincides with the Laplace sequence considered by L. Godeaux.

C. C. Hsiung (Madison, Wis.)

2264:

Ichijô, Yoshihiro. On some properties of plane curves in Riemann spaces. J. Gakugei Tokushima Univ. Math. **9** (1958), 25–30.

The author writes down the differential equations for a plane curve contained in a V_n immersed in a V_m and concludes that (i) if a plane curve in V_n in V_m is a Darboux line in V_n in V_m [see Ichijô, same J. **8** (1957), 27–32; MR **20** #3569], it is an asymptotic curve or geodesic circle; (ii) if a V_n is a totally geodesic subspace in V_m, then a plane curve in V_m is a plane curve in V_n and a Darboux line in V_n in V_m. He presents next a necessary and sufficient condition that a plane curve in V_{n+1} be a plane curve in a V_n in the V_{n+1} and the theorems (iii) if a plane curve of V_{n+1} is contained in a perfectly totally umbilic hypersurface, it is a geodesic circle of V_{n+1}; (iv) a sufficient condition that every plane curve in V_{n+1} be a plane curve in V_n when it is contained in V_n, is that V_n be either totally geodesic or perfectly totally umbilic in V_{n+1}. *A. Schwartz* (New York, N.Y.)

2265:

Huang, Cheng-chao. On spaces of constant curvature and the totally umbilical hypersurfaces. Acta Math. Sinica **8** (1958), 490–495. (Chinese. English summary)

This paper is concerned with the systems of umbilic hypersurfaces (i.e., hypersurfaces whose first and second fundamental forms are proportional at every point) in a Riemannian space. For an umbilic hypersurface of constant curvature, the author defines a simple invariant, called U curvature. It is proved that if a Riemannian space V_n of dimension $n > 3$ has three systems of mutually orthogonal umbilic hypersurfaces of constant curvature, then V_n is of constant curvature K and the U curvature of each of these hypersurfaces is equal to K. This theorem no longer holds when $n = 3$. The conclusion is nevertheless valid if, moreover, the U curvatures of the hypersurfaces of the three systems are assumed to be all equal. Riemannian spaces with two systems of mutually orthogonal umbilic hypersurfaces of constant curvature are also discussed. *H. C. Wang* (Evanston, Ill.)

2266:

Solodovnikov, A. S. Uniqueness of a maximal K-expansion. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 173–179. (Russian)

In the author's paper on projective transformations of Riemannian spaces [same Uspehi **11** (1956), no. 4 (70), 45–116; MR **18**, 930] the exceptional metrics were described in terms of maximal K-decompositions of the line element into sums of elements of constant curvature. Here the uniqueness of these decompositions is proved.

L. W. Green (Minneapolis, Minn.)

2267:

Laugwitz, Detlef. Über eine Vermutung von Hermann Weyl zum Raumproblem. Arch. Math. **9** (1958), 128–133.

Let E be a hypersurface in an n-dimensional vector space given by $\varphi(\xi^i) = 1$; $\varphi(\lambda\xi^i) = |\lambda|\varphi(\xi^i)$, $\varphi(\xi^i) > 0$ for $\xi \neq 0$. This induces a metric on a differentiable manifold by the formula

$$f(x, dx) = \varphi[B_k^i(x)dx^k],$$

where $B_k^i(x)$ is an arbitrary tensor field with $|B_k^i| \neq 0$. This metric is called a realization of E. E is said to have "property P" if for every realization of E there exists a length-preserving affine connection. H. Weyl has conjectured that property P is equivalent to the statement that φ is a positive definite quadratic form; i.e., that $f(x, dx)$ is Riemannian. It is proved that if E is a bounded, star-shaped, convex surface which has property P, then f is Riemannian. Moreover, if E has property P, it permits an r-parameter continuous group of affinities with $r \geq 1$.

C. B. Allendoerfer (Seattle, Wash.)

2268:

Auslander, Louis. On the sheeted structure of compact locally affine spaces. Michigan Math. J. **5** (1958), 163–168.

Let M be a compact n-dimensional manifold with a flat

affine connexion, and let Γ be the fundamental group of M. The affine connexion imbeds Γ in the group of affine transformations of affine n-space, and hence defines the subgroup T of pure translations in Γ. Let $h(\Gamma)=\Gamma/T$ = the holonomy group of Γ. Three theorems are proved. The first states that if T is free abelian with $s\geq 1$ generators and $h(\Gamma)$ has no elements of finite order, then M is a fiber bundle over a compact manifold X with a flat affine connexion, and with the s-dimensional torus as fiber. The other two theorems state, under weaker assumptions, that there exists an X and a mapping of M onto it, with fewer properties.

W. Ambrose (Cambridge, Mass.)

2269:

Postelnicu, Tiberiu. Étude des espaces A_2 à connexion affine linéaire, localement euclidiens. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 279–301. (Romanian. Russian and French summaries)

A space $A_n(x^1, x^2, \cdots, x^n)$ with affine connection given by the components $\Gamma_{jk}^{i}(x^1, x^2, \cdots, x^n)$ is said to be locally euclidean if the torsion tensor, $T_{jk}^{i}=\Gamma_{jk}^{i}-\Gamma_{kj}^{i}$, and the curvature tensor are zero. In this case there exists a 'cartesian' coordinate system $u^1, u^2, \cdots, u^n$ in which the components of the connection are zero, and the auto-parallel lines are, therefore, straight lines. If A_n is locally euclidean and coincides with the euclidean space E_n, the correspondence being one-to-one, A_n is said to be 'globally' equivalent to E_n. The transformation $u^i=u^i(x^1, x^2, \cdots, x^n)$ is then topological.

The problem of equivalence in the large of locally euclidean A_n with the corresponding euclidean spaces E_n has been considered by Vranceanu in its general form and then in the particular case when the Γ_{jk}^{i} are constants.

The present paper is a study of locally euclidean $A_2(x^1, x^2)$ for which the Γ_{jk}^{i} are linear in the x^1, x^2. It is shown that there are essentially six different cases, for which canonical forms are given. The problem of equivalence in the large of these A_2 with the euclidean plane E_2 is then considered. It turns out that in two cases such an equivalence actually exists. In the remaining cases the A_2 is globally equivalent with (1) the half plane $u^1>0$, (2) the first quadrant, etc.

R. Blum (Saskatoon, Sask.)

2270:

Postelnicu, Tiberiu. Classification des transformations ponctuelles associées aux espaces A_2 à connexion affine linéaire. Com. Acad. R. P. Romîne **8** (1958), 13–18. (Romanian. Russian and French summaries)

In a previous paper [reviewed above] the author has shown that there exist essentially 6 different two-dimensional spaces with linear affine connection A_2 which are locally euclidean (i.e., $R_{ijk}^{h}=0$).

In the present paper the author determines the projective connections corresponding to these 6 cases and classifies them according to the 'characteristic directions'. These are the directions in a point P (in general 3 in number) along which the auto-parallel line has a point of inflection in P. It turns out that the A_2 for which there is no equivalence in the large with the euclidean plane have 3 or 2 distinct characteristic directions, whereas for the A_2 which are equivalent in the large with the euclidean plane the number of distinct characteristic directions is either 2 or 1.

R. Blum (Saskatoon, Sask.)

2271:

★**Lichnerowicz, André. Groupes d'holonomie.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 334–347. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

A summary of the basic facts about holonomy groups. Starting with the definition of a connexion it defines the holonomy group and restricted holonomy group. Then it passes to affine connexions and parallel translation, discussing for these the relation between holonomy and curvature. Then it treats reducibility and factoring of the holonomy groups, and closure. It concludes by discussing the holonomy of pseudo-kahlerian manifolds and homogeneous spaces, including Hermitian homogeneous spaces.

W. Ambrose (Cambridge, Mass.)

2272:

Nevanlinna, F.; und Nevanlinna, R. Über die Integration eines Tensorfeldes. Acta Math. **98** (1957), 151–170.

This article is an improved version of an article published earlier [R. Nevanlinna, Ann. Acad. Sci. Fenn. Ser. A. I. no. **219** (1956); MR **18**, 295]. The authors use a coordinate-free method with the following notation: (1) $s^{p+1}(x_0, x_1, \cdots, x_{p+1})$ is the set of interior and boundary points of an oriented $(p+1)$-simplex in a Euclidean m-space R_x^m; (2) k_i are vectors in a linear space with a basis

$$h_1 = x_1-x_0, \quad h_2 = x_2-x_0, \quad \cdots, \quad h_{p+1} = x_{p+1}-x_0;$$

(3) $y=A(x)k_1k_2\cdots k_p$ is a linear alternating form in the arguments $k_1, k_2, \cdots, k_p$ defined on s^{p+1} and with values in R_y^n; (4) rot $A(x)k_1k_2\cdots k_{p+1}$ is the exterior differential form of the form $A(x)k_1k_2\cdots k_p$; (5) G_x^m is an open star-like region about the point O of the space R_x^m (a region is called star-like if with a point x the segment Ox belongs to the region). The article gives a detailed, systematic explanation and investigation of: (1) the integral theorem asserting that

$$\int_{\partial s^{p+1}} A(x)d_1xd_2x\cdots d_px = \int_{s^{p+1}} \text{rot } A(x)d_1xd_2x\cdots d_{p+1}x,$$

where ∂s^{p+1} is the set of boundary points s^{p+1} and $A(x)k_1k_2\cdots k_p$ is a continuously differentiable alternating form (the authors call it Stokes's rule); (2) the equation rot $Y(x)=A(x)$ and its solution, where

$$y = A(x)d_1xd_2x\cdots d_px$$

is an alternating differential defined on G_x^m and $Y(x)$ is the desired alternating differential form of rank $(p-1)$.

Especial attention is given to possible weakening of the restrictions on $A(x)$. The authors succeed in extending the definition of rot $A(x)$ to certain non-differentiable and even non-continuous forms $A(x)$ and then in showing that the existence of a solution of the equation rot $Y=A$ is already guaranteed by the conditions of continuity of $A(x)$ and of existence and vanishing of rot $A(x)$.

P. M. Oloničev (RŽMat **1958** #8257)

2273:

Hangan, Teodor. Automorphismes infinitésimaux et holonomie. C. R. Acad. Sci. Paris **247** (1958), 411–412.

A connexion is considered on a principal fiber bundle E, with 1-form ω. An infinitesimal automorphism is

defined to be a vector field ξ on E for which the Lie derivative of ω relative to ξ is 0. Let G denote the group of E and L its Lie algebra. Let L' be any Lie algebra of right invariant infinitesimal automorphisms. For each $z \in E$ the subgroup of G corresponding to the subalgebra of L generated by all the $\omega(\xi(z))$, $\xi \in L'$, is called the Kostant group of L' at z and denoted by $K_z(L')$. The following two theorems are stated: (1) $K_z(L')$ is contained in the normalizer of the infinitesimal holonomy group at z; (2) under a further condition the infinitesimal holonomy group at z equals the local holonomy group at z and is contained in $K_z(L')$. *W. Ambrose* (Cambridge, Mass.)

2274:

Lelong-Ferrand, Jacqueline. Sur l'application des méthodes de Hilbert à l'étude des transformations infinitésimales d'une variété différentiable. Bull. Soc. Math. Belg. **9** (1957), 59–73.

This paper contains the detailed proofs of results announced earlier [C. R. Acad. Sci. Paris **245** (1957), 1585–1588; MR **20** #2023]. *L. Auslander* (Bloomington, Ind.)

2275:

Devi Singh, K.; and Mishra, R. S. Infinitesimal deformations of a Riemannian space. Riv. Mat. Univ. Parma **7** (1956), 79–88.

λ désignant un champ de vecteurs sur la variété riemannienne V_n, de tenseur métrique g_{ij}, une déformation infinitésimale est définie par $\bar{x}^i = x^i + \varepsilon\lambda^i(x)$, et le vecteur $\bar{u}$, déformé de u, par $\bar{u}^i = u^i + \varepsilon(u^r,_k\lambda^k - u^h\lambda^r,_h)$; et l'on prend $\bar{g}_{ij} = g_{ij} + \varepsilon(\lambda_{i,j} + \lambda_{j,i})$ pour tenseur métrique au point $\bar{x}$ sur la variété déformée $\bar{V}_n$. Supposant λ de Killing, les A. étudient de divers problèmes, entre autres les conditions (a) pour que $u \to \bar{u}$ transforme la tangente au point x à une courbe C, en la tangente au point $\bar{x}$ à la courbe déformée; (b) pour que, dans la déformation, les tangentes à une courbe se déplacent parallèlement. *J. Lelong* (Paris)

2276:

Kobayashi, Shoshichi. Fixed points of isometries. Nagoya Math. J. **13** (1958), 63–68.

ξ désignant un champ de vecteurs de Killing sur une variété riemannienne M, l'A. établit de diverses propriétés des composantes connexes de l'ensemble F des points où ξ s'annule; entre autres, si M est compacte, le nombre d'Euler de M est la somme des nombres d'Euler de ces diverses composantes. Ces résultats s'étendent au cas d'un groupe infinitésimal abélien quelconque d'isométries. *J. Lelong* (Paris)

2277:

Samelson, Hans. On curvature and characteristic of homogeneous spaces. Michigan Math. J. **5** (1958), 13–18.

Let G be a compact connected group, U a closed subgroup of G and $M = G/U$ the homogeneous space formed by the left cosets of U, and let p be the natural projection of G onto M. The group G admits Riemannian metrics that are invariant under left and right translations; let such a metric be chosen.

Let x be a point in M and let X be a tangent vector of M at x. Let x' be a point of G with $p(x') = x$ and let X' be a tangent vector of G at x', orthogonal to the coset $x' \cdot U$ at x', with $dp(X') = X$. Then the norm $|X|$ of X is defined to be equal to the norm $|X'|$ of X'. This metric is called induced Riemannian metric in M.

In the first part of the present paper, the author proves the following very interesting theorem: All values of the sectional Riemannian curvature of M, in the induced Riemannian metric, are non-negative. The theorem has been proved earlier by É. Cartan for symmetric spaces.

In the second part of the paper, the author gives a new proof for the following important theorem: The Euler-Poincaré characteristic $x(M)$ of $M = G/U$ is non-negative; it is positive exactly if U is of maximal rank. The theorem has been proved earlier by H. Hopf, A. Weil and the author himself. *K. Yano* (Tokyo)

2278:

Toponogov, V. A. Riemann spaces with curvature bounded below. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 87–130. (Russian)

The principal theorem proved in the paper is that if the curvature at every point and for every 2-direction in a twice continuously differentiable Riemannian manifold R_k^m with a complete metric is not smaller than k, then every triangle composed of geodesics on R_k^m has angles which are not smaller than the corresponding angles of the triangle with sides of the same length in a surface of constant curvature k. From this follow the theorems that if $k > 0$, then the sum of length of sides of any triangle on R_k^m is not greater than $2\pi/\sqrt{k}$; and that if in a R_k^m ($k > 0$) there exists a triangle with perimeter $2\pi/\sqrt{k}$, then R_k^m is isometric with an m-dimensional sphere.

The paper consists of 4 chapters. The first chapter explains the definitions of the fundamental concepts, terminologies and notations, and formulates some known results which are used in the paper. In the second chapter, fundamental lemmas are proved, viz., the lemma on a limiting angle and the lemma on a small triangle. The principal theorem is demonstrated in the third chapter. The last chapter applies the obtained results to a convex m-dimensional subspace in an $(m+1)$-dimensional euclidean space. An outline of these results was published before [Dokl. Akad. Nauk SSSR **115** (1957), 674–676; MR **19**, 979]. *A. Kawaguchi* (Sapporo)

2279:

Zalgaller, V. A. Isometric imbedding of polyhedra. Dokl. Akad. Nauk SSSR **123** (1958), 599–601. (Russian)

A polyhedron P^n made up of simplexes from a space of constant curvature R^n is said to be isometrically imbedded in R^n if it has a decomposition into simplexes which may be isometrically mapped into R^n with preservation of coincidences. Overlappings and self-intersections are allowed. (For example, map the vertices of a triangle into a single point and the sides, folded at their bisectors, into a ray from this point.) Theorem: For $n = 1, 2, 3, 4$ any polyhedron P^n made up of simplexes from a space R^n of constant curvature may be isometrically imbedded in R^n. The proof is given for the case $n = 2$, and a proof by induction modulo a certain construction for arbitrary n. The possibility of the construction is proved for $n = 3$ and stated for $n = 4$. *L. W. Green* (Minneapolis, Minn.)

2280:

Grotemeyer, K. P. Zur Flächentheorie im Grossen: Über die Abbildung durch parallele Normalen. Arch. Math. **9** (1958), 117–122.

Let $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ be two bounded simply connected surfaces of class C^2 and of positive curvature. They are in 1-1 correspondence such that corresponding points $\mathfrak{x}$ and $\hat{\mathfrak{x}}$ have the same normal vector ξ. The boundary $\partial\mathfrak{R}$ of $\mathfrak{R}$ is required to belong to C^2 and to be congruent to $\partial\hat{\mathfrak{R}}$ by translation. A vector $\mathfrak{e}$ is assumed to exist such that no straight line parallel to $\mathfrak{e}$ intersects $\mathfrak{R}$ or $\hat{\mathfrak{R}}$ at more than one point. Let K and H [$\hat{K}$ and $\hat{H}$] denote the Gaussian and mean curvatures of $\mathfrak{R}$ [of $\hat{\mathfrak{R}}$]. Suppose the function $F(U, V)$ is defined, continuous and monotonic in the same sense in both variables; $U>0$, $U^2 \geqq 4V>0$. Let

$$F\left(\frac{2H}{K}, \frac{1}{K}\right) = F\left(\frac{2\hat{H}}{\hat{K}}, \frac{1}{\hat{K}}\right)$$

at corresponding points of $\mathfrak{R}$ and $\hat{\mathfrak{R}}$. Then $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ are congruent by translation.

Outline of proof: The derivatives $\mathfrak{v}_i$ of $\mathfrak{v}=\mathfrak{x}-\hat{\mathfrak{x}}$ are expressed through those of $\mathfrak{x}$, say (1) $\mathfrak{v}_i = \Pi_i^k \mathfrak{x}_k$. The author proves $\|\Pi_i^k\| \leqq 0$, equality holding only if $\Pi_i^k = 0$. Since $\mathfrak{v}=\text{const}$ on $\partial\mathfrak{R}$, (1) implies

$$\int_{\mathfrak{R}} \|\Pi_i^k\| \cdot \mathfrak{e}\xi \cdot do = \tfrac{1}{2} \oint_{\partial\mathfrak{R}} (\mathfrak{e}\mathfrak{v})\mathfrak{v} ds = 0.$$

As $\mathfrak{e}\xi$ is never positive or never negative, this implies $\|\Pi_i^k\| = 0$ on $\mathfrak{R}$ and hence $\Pi_i^k = 0$ and $\mathfrak{v}=\text{const}$.

The corresponding result for the boundaries $\mathfrak{F}$ and $\hat{\mathfrak{F}}$ of convex bodies was obtained by A. D. Alexandroff [Dokl. Akad. Nauk SSSR **22** (1939), 99–102] if $\mathfrak{F}$, $\hat{\mathfrak{F}}$, and F are analytical and by Hartman and Wintner [Amer. J. Math. **75** (1953), 298–334; MR **14**, 1119] if $\mathfrak{F}$, $\hat{\mathfrak{F}} \in C^3$, $F \in C^2$.

P. Scherk (Boulder, Colo.)

2281:

Grotemeyer, K. P. Zur Flächentheorie im Grossen. II. Einfache Bemerkungen zur Poincaréschen Indexmethode. Arch. Math. **9** (1958), 382–388.

A unified approach to various problems of surface theory is based on the following theorem. On a surface (local, with parameters u^1, u^2) of class C^2 let symmetric tensors S^{ik} and L_{ik} be defined and of class C^1. If S^{ik} is positive definite, $S^{ik}L_{ik}=0$, $\varepsilon^{ik}L_{ik||s}=0$, then either $L_{ik}\equiv 0$ or the places where $L_{ik}=0$ are isolated and the curve system $L_{ik}du^i du^k = 0$ has at these points a negative index. Hence, if F is closed and of genus 0, then $L_{ik}\equiv 0$.

The following well-known theorems are easy consequences: The monotypy of closed surfaces of class C^2 with positive curvature within this class, i.e., that two such surfaces, if intrinsically isometric, are congruent. The rigidity of the same surfaces. The fact that a closed surface with genus zero and constant mean curvature is a sphere. Finally, properties of the so-called "Drehriss" of a surface with positive curvature are discussed.

H. Busemann (Cambridge, Mass.)

2282:

Rembs, E. Flächen mit gleichen Summen der Hauptkrümmungsradien. Arch. Math. **8** (1957), 469–471.

Suppose corresponding points on the surfaces $\mathfrak{x}(u, v)$ and $\bar{\mathfrak{x}}(u, v)$ have the same spherical images and the same sums of the principal radii of curvature. Then $\mathfrak{x}-\bar{\mathfrak{x}}$ is a minimal surface. If in addition $\mathfrak{x}$ and $\bar{\mathfrak{x}}$ are convex and closed, this implies $\mathfrak{x}-\bar{\mathfrak{x}}=\text{const.}$, i.e., Christoffel's theorem that $\mathfrak{x}$ and $\bar{\mathfrak{x}}$ are congruent. The argument is extended to the case that $\mathfrak{x}$ and $\bar{\mathfrak{x}}$ are non-closed convex surfaces such that a suitable transformation maps the boundary of $\mathfrak{x}$ onto that of $\bar{\mathfrak{x}}$. {The author proves his initial remark by introducing isothermic coordinates on the common spherical image and applying Codazzi's equations. Actually it is a corollary of the formulas

$$K = \frac{eg-f^2}{LN-M^2}, \quad 2H = \frac{Lg-2Mf+Ne}{LN-M^2},$$

which follow readily from the formulas (30), (31), and (87) on pp. 92 and 104 of Blaschke's *Vorlesungen über Differentialgeometrie*, vol. I, 3rd ed., Springer, Berlin, 1930.}

P. Scherk (Boulder, Colo.)

2283:

Švec, Alois. Remarque sur la déformation projective des hypersurfaces développables. Časopis Pěst. Mat. **84** (1959), 50–52. (Czech. Russian and French summaries)

In this note the author gives a geometric interpretation of the É. Cartan's equations for the deformations of a developable hypersurface in projective four space [Ann. Sci. École Norm. Sup. (3) **37** (1920), 259–356].

J. J. Kohn (Waltham, Mass.)

2284:

Stoka, Marius I. Geometria integrale in uno spazio euclideo E_n. Boll. Un. Mat. Ital. (3) **13** (1958), 470–485. (English summary)

Let $\{V\}$ be a set of geometrical elements in the euclidean space E_n which transform transitively by a group of transformations G which operates in E_n. The transformations of G which leave invariant a fixed element V form a subgroup g. If the homogeneous space G/g possesses an invariant measure and if this measure is the same for any G, then the set $\{V\}$ is called measurable. The author gives some properties of measurable sets and many examples of measurable and non-measurable sets on the plane.

L. A. Santaló (Buenos Aires)

2285:

Stoka, Marius I. Mesure des familles de variétés de l'espace E_3. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 547–558. (Romanian. Russian and French summaries)

This is an extension of the paper reviewed above, giving several new examples of measurable and non-measurable sets of geometrical elements in the euclidean 3-space. For instance: (a) Sets of pairs of lines are measurable (i.e., they have an invariant measure which is the same with respect to any linear group); (b) sets of pairs of planes are non-measurable (i.e., they have different invariant measure with respect to different linear groups).

L. A. Santaló (Buenos Aires)

2286:

Rešetnyak, Yu. G. The method of orthogonal projections in the theory of curves. Vestnik Leningrad. Univ. **12** (1957), no. 13, 22–26. (Russian. English summary)

Some methods of I. Fáry [Bull. Soc. Math. France, **77**

(1949), 128–138; MR **11**, 393] are extended to a wider class of curves in E^3 by polygonal approximation. Formulas are given for total curvature and total torsion in terms of integrals over the orthogonal projections. A priori estimates on the inscribed polygons also yield inequalities like

$$s(K) \leq \frac{2}{\pi} d(K)[\chi(K)+\pi],$$

where s, d, χ are length, diameter, and total curvature, respectively, of the curve K. Generalizations to Lobačevskiĭ space are indicated.

L. W. Green (Minneapolis, Minn.)

2287:

Haupt, Otto. Streng-konvexe Bogen und Kurven in der direkten Infinitesimalgeometrie. Arch. Math. **9** (1958), 110–116.

The author first defines ordinary differentiability of a curve in a projective space R_n of dimension n, and then extends the definition [M. Barner, Abh. Math. Sem. Univ. Hamburg **20** (1956), 196–215; MR **19**, 60] of a strongly convex curve of class $C^{(n)}$ in R_n to ordinarily differentiable closed arcs and curves free from $(n-2, n)$-secants in a broader sense. Concerning these arcs and curves, a theorem containing several parts, together with a brief sketch of its proof, is given. The special case of a typical part of the theorem can be stated as follows: If a strongly convex arc in R_n has k $(\geqq n)$ points in common with a hyperplane, then the arc has at least $k-n$ points with stationary osculating hyperplane.

C. C. Hsiung (Bethlehem, Pa.)

2288:

Mirguet, Jean. Sur une distinction entre les paratingentes secondes d'une surface. Rend. Sem. Mat. Univ. Padova **27** (1957), 245–252.

Cette note développe une note précédente [C. R. Acad. Sci. Paris **245** (1957), 402–404; MR **19**, 877] et contient en outre quelques considérations sur les orthosurfaces telles qu'en chaque point l'intersection de chaque plan totalement paratingent et du paratingent non libre n'admette aucun sous-ensemble infini continu, en particulier se réduise à une droite ou deux droites. [Cf. ibid. **245** (1957), 488–490; MR **19**, 878.] *Chr. Pauc* (Nantes)

2289:

Bouligand, Georges. Modèles euclidiens de métriques variationnelles. C. R. Acad. Sci. Paris **246** (1958), 345–347.

2290:

Coz, Marcel. Une classe d'équations intégrales liées à des métriques variationnelles à indicatrices centrées. C. R. Acad. Sci. Paris **246** (1958), 877–880.

2291:

Bouligand, Georges. Observations au sujet de la Note de M. Marcel Coz. C. R. Acad. Sci. Paris **246** (1958), 880.

2292:

Coz, Marcel. Équations intégrales subordonnées à une condition de prolongement analytique. C. R. Acad. Sci. Paris **246** (1958), 2715–2717.

2293:

Bouligand, Georges; et Coz, Marcel. Modèles géométriques et équations intégrales. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 635–646.

2294:

Bouligand, Georges. A propos des chances restreintes d'objectivité physique de métriques variationnelles. C. R. Acad. Sci. Paris **247** (1958), 1155–1156.

The author suggests modifications possible in the treatment of the Schwarzschild problem in relativity theory based on variational metrics which he has developed jointly with M. Coz [#2293 above] and considers the possible objectivity, i.e. correspondence to the physical situation, of the suggested metric.

W. M. Boothby (Evanston, Ill.)

PROBABILITY

See also 2038, 2190, 2386, 2410.

2295:

Dubins, Lester E. Conditional probability distributions in the wide sense. Proc. Amer. Math. Soc. **8** (1957), 1088–1092.

The following result is proved. $(U, \mathscr{U}, u)$ is a probability space, $\mathscr{U}_0$ is a sub σ-field of $\mathscr{U}$ and X is a locally compact Hausdorff space whose one point compactification is metrizable. If y is a random variable defined on U and taking values in X, then y possesses a wide sense conditional distribution relative to $\mathscr{U}_0$. That is, there exists a function $p(Y, w)$ where Y ranges over the Baire subsets of X and w ranges over U such that (i) for each fixed w, p is a probability measure over the Baire subsets of X; (ii) for each Y, p is a function measurable relative to $\mathscr{U}_0$; and (iii) $\int_A p(Y, w)\, du(w) = u(A \cap y^{-1}(Y))$ for every $A \in \mathscr{U}_0$ and every Baire subset Y of X. This result generalizes a well-known one of Doob in which the range space X of the random variable is an n-dimensional Euclidean space.

G. Kallianpur (East Lansing, Mich.)

2296:

Buchner, P. Überraschende Ergebnisse der Wahrscheinlichkeitsrechnung. Elem. Math. **13** (1958), 75–78.

The author gives examples of problems in the theory of probability (related to the arcsine law) the solutions of which are contrary to one's intuition.

A. H. Copeland, Sr. (Ann Arbor, Mich.)

2297:

Maravall Casesnoves, Dario. The addition of random isotropic vectors in an n-dimensional space. Trabajos Estadist. **9** (1958), 183–202. (Spanish. English summary)

A density function (or distribution function) is called isotropic if it is spherically symmetric. The author studies: (i) an isotropic distribution in Euclidean 2-space as a function of the coordinates (x, y) and the distance r from

the origin to one of the coordinates; (ii) the marginal density of the marginal distribution of r when the marginal distribution of x is known; (iii) the generalisation of the above to Euclidean n-space when the marginal distribution of a given coordinate, say x_t, is assumed to be normal, and it is shown that the n coordinates $x_1, x_2, \cdots, x_n$ are normal and independently distributed; (iv) some cases of addition of isotropic vectors.

A. T. Bharucha-Reid (Eugene, Ore.)

2298:

Baxter, Glen. An operator identity. Pacific J. Math. 8 (1958), 649–663.

Let $\{X_k\}$ be a sequence of independent, identically distributed random variables and denote by $\{S_n\}$ the sequence of partial sums. Starting with combinatorial ideas, F. Spitzer [Trans. Amer. Math. Soc. **82** (1956), 323–339; MR **18**, 156] obtained the identity

$$(*) \qquad \sum_{n=0}^{\infty} \varphi_n(t)s^n = \exp\left\{\sum_{k=1}^{\infty} s^k\psi_k(t)/k\right\},$$

where φ_n and ψ_k are the characteristic functions of $\max(0, S_1, \cdots, S_n)$ and $\max(0, S_k)$, respectively. The author applies analytical methods, generalises the above to an identity between operators and applies the result to stationary Markov chains. *A. Dvoretzky* (Jerusalem)

2299:

Wendel, J. G. Spitzer's formula: a short proof. Proc. Amer. Math. Soc. **9** (1958), 905–908.

The author reverses the procedure of F. Spitzer in the paper mentioned in the preceding review. He proves a generalised version of (*) analytically, and then deduces from (*) the fundamental combinatorial lemma of Spitzer.

A. Dvoretzky (Jerusalem)

2300:

Linnik, Yu. V. General theorems of the factorization of infinitely divisible laws. Teor. Veroyatnost. i Primenen. **4** (1959), 55–85. (Russian. English summary)

The first part appeared in **3** (1958), 3–40 of the same journal. Here the detailed proof of the sufficiency part of the factorization theorem quoted in the review [MR **20** #4883] is given (for the case of bounded Poisson spectrum). The author also obtains the form of the components of an infinitely divisible law with bounded Poisson spectrum, rational to the left of a given point.

A. Dvoretzky (Jerusalem)

2301:

Kahane, Jean-Pierre. Sur le recouvrement d'un cercle par des arcs disposés au hasard. C. R. Acad. Sci. Paris **248** (1959), 184–186.

Let T be a circle of unit circumference and let l_n $(n=1, 2, \cdots)$ be a decreasing sequence of positive numbers with $l_1 < 1$. Let arcs L_n of lengths l_n be placed "at random" on T. Continuing work of the reviewer [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 199–203; MR **18**, 75], the author proves that $\lim\sup_{t\to\infty} (\log t)^{-1} \sum_{l_n>1/t} l_n > 1$ implies $T = \bigcup L_n$ with probability 1. He also shows that the 1 on the right side of the inequality cannot be replaced by any smaller number. *A. Dvoretzky* (Jerusalem)

2302:

Praporgescu, N. Sur les probabilités en chaîne. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 439–480. (Romanian. Russian and French summaries)

The author uses a most powerful method in order to investigate asymptotical problems concerning Markov processes and processes in chains with complete connections. The method is based on solving the functional equations which were given by Onicescu and Mihoc for different problems concerning these processes [Acad. Roum. Études Recherches **14** (1943); MR **10**, 311].

The inversion used by the author constitutes an application of a more general theory given by himself on a class of integral equations [Acad. R. P. Romîne Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 549–616; MR **18**, 808].

O. Onicescu (Bucharest)

2303:

Zítek, František. Équations différentielles stochastiques. Czechoslovak Math. J. **8** (**83**) (1958), 465–472. (Russian summary)

This work is based upon the following definition of derivative. Let $x(t)$ be a stochastic process with independent increments and with the cumulant generating function $\psi_x(t; z)$, and let $y(t)$ be a stochastic process with independent values for different t's and with the cumulant generating function $\psi_y(t; z)$. Then $y(t)$ is said to be the derivative of $x(t)$ if

$$\frac{\partial}{\partial t}\psi_x(t; z) = \psi_y(t; z);$$

this is denoted by $Dx(t) \sim y(t)$. This differential equation is studied, as is the equation "with errors" $\delta x(t) \sim F(t, dt)$, which is interpreted as

$$|\psi_x(t+h; z) - \psi_x(t; z) - \psi_F(t, h; z)| = O(h)$$

uniformly in every bounded z-interval. Conditions are given for integrability. *U. Grenander* (Stockholm)

2304:

Zítek, František. Sur l'intégrabilité d'une équation différentielle stochastique. Czechoslovak Math. J. **8** (**83**) (1958), 473–482. (Russian summary)

The author continues his investigation of stochastic differential equations [see also the paper reviewed above] $\delta x(t) \sim yg(dt)$, where $g(h)$ is non-decreasing, $g(0)=0$, $g(+\infty)=+\infty$. The results are, essentially, the following. A necessary and sufficient condition for the equation ("without error") to have a non-trivial solution is that $g(h) = ah^{1/\alpha}$, $a>0$, $0<\alpha\leq 2$, and that y has a certain stable distribution. For the equation "with error" the distribution of y should belong to the domain of attraction of a certain non-singular stable distribution.

U. Grenander (Stockholm)

2305:

Mešalkin, L. D. Limit theorems for Markov chains with a finite number of states. Teor. Veroyatnost. i Primenen. **3** (1958), 361–385. (Russian. English summary)

Consider a sequence of $s \times s$ stochastic matrices $P(n)$. The nth Markov chain has initial state 1 and stationary transition matrix $P(n)$. Let $\mu(n)$ be the number of passages

through the state 1 in n steps in the nth chain. A number of complicated distributions are given through their characteristic functions such that if the distribution of $\alpha(n)[\mu(n)-n\theta(n)]$ has a limit for arbitrary normalizing sequences $\alpha(n)$ and $\theta(n)$, then it must be one of these. The proofs of the results depend on matrix theory and some examples are given. For $s=2$ see Dobrušin [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 291–330; MR **15**, 329].

K. L. Chung (Syracuse, N.Y.)

2306:

Volkov, I. S. On the distribution of sums of random variables defined on a homogeneous Markov chain with a finite number of states. Teor. Veroyatnost. i Primenen. **3** (1958), 413–429. (Russian. English summary)

"Local and integral theorems are established for a non-periodic case. The results are given in the form of asymptotic expansions taking into account various possible values of the sums under consideration." (Author's summary)

K. L. Chung (Syracuse, N.Y.)

2307:

Rosenblatt, M. Functions of a Markov process that are Markovian. J. Math. Mech. **8** (1959), 585–596.

Given a function f mapping the positive integers into themselves and a Markov chain $x(t)$: $t\geqq 0$ with positive integral states and decent transition probabilities $p_{ij}(t)$: $i, j=1, 2, 3$, etc., $t\geqq 0$, the author shows that $f[x(t)]$: $t\geqq 0$ is Markovian for each initial distribution of the chain if and only if each of the ordinate sets $f^{-1}(k)$ satisfies either the condition (1) $p_{ij}(t)=0$, $i\notin f^{-1}(k)$, $j\in f^{-1}(k)$, $t\geqq 0$, or the condition (2) $\sum_{j\in f^{-1}(k)} p_{ij}(t)$ is a function of t and of the class $f^{-1}(f(i))$ alone. The same problem is solved for chains with integral time and a general state space. The result is more complicated to state but the same in spirit.

H. P. McKean, Jr. (Cambridge, Mass.)

2308:

Čerkasov, I. D. The method of separation in the theory of Markov processes. Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), 23–32. (Russian. English and Romanian summaries)

It is known that continuous Markov processes may be described by

$$\frac{\partial f}{\partial t}+A^i(t,x)\nabla_i f+B^{ij}(t,x)\nabla_i\nabla_j f=0,$$

where $f(t,x;\pi,\xi)$ is the probability density function and A^i and B^{ij} are defined in the n-dimensional space R_n with metric $ds^2=dx_1^2+\cdots+dx_n^2$. The author investigates the problem concerning necessary and sufficient conditions that

$$f(t,x;\pi,\xi)=\prod_{r=1}^{s} f^r(t,x^{(r)};\pi,\xi^{(r)}),$$

where $(x^{(r)})=(x_{k_{r-1}+1},\cdots,x_{k_r})$ with $r=1,\cdots,s$ and f^r is the probability density of certain processes in $R_{(r)}=(x^{(r)})$. Necessity is proved under the hypothesis: R_n is a limited domain of euclidean space, and A^i and B^{ij} are infinitely differentiable functions. The author illustrates the invariability of the fundamental relations of Markov process by an example.

H. P. Edmundson (Pacific Palisades, Calif.)

2309:

Dynkin, E. B. One-dimensional continuous strong Markov processes. Teor. Veroyatnost. i Primenen. **4** (1959), 3–54. (Russian. English summary)

The author finds the infinitesimal operator of any continuous parameter Markov stochastic process whose state space is a linear interval, which has stationary transition probabilities, enjoys the strong Markov property, and has continuous sample functions. The main results have already been announced [Dokl. Akad. Nauk SSSR **105** (1955), 405–408; MR **17**, 866]. They apply to a slightly wider class of processes than those considered by Feller [Ann. of Math. (2) **60** (1954), 417–436; **61** (1955), 90–105; MR **16**, 488, 824] in solving the same problem. Feller's treatment was purely analytical, whereas Dynkin bases his on a detailed study of the process sample functions.

J. L. Doob (Urbana, Ill.)

2310:

Kramer, H. P. Symmetrizable Markov matrices. Ann. Math. Statist. **30** (1959), 149–153.

The author is concerned with Markov processes having a finite set of states labelled as $(1, 2, \cdots, N)$, so that the state probabilities $p_i(t)$ satisfy equations

$$\frac{dp_i}{dt}=\sum_{j=1}^{N} Q_{ij}p_j(t)$$

(note that Q_{ij} is the transition-rate from state j to state i, and not conversely). If μ is a sequence of N positive numbers $\mu_1, \cdots, \mu_N$, the author calls the matrix Q $l_2(\mu)$-symmetric when $\mu_i Q_{ij}=\mu_j Q_{ji}$, and his theorem 1 asserts that, when $Q_{ij}\neq 0$ implies $Q_{ji}\neq 0$, then Q will be $l_2(\mu)$-symmetric for some μ if and only if a certain "loop condition" holds. This loop condition was introduced by A. N. Kolmogorov [Math. Ann. **112** (1936), 155–160] in connexion with the analogous problem for chains, and was further employed by E. Reich [Ann. Math. Statist. **28** (1957), 768–773; MR **19**, 1203]. The sequence μ is unique up to a multiplying factor when the loop condition is satisfied, and under an irreducibility condition {which appears to the reviewer to be unnecessarily strong} it is shown that then $\pi_i\equiv p_i(\infty)=\mu_i^{-1}/\sum \mu_n^{-1}$, and that

$$(*)\qquad e^{-\beta t}\|p(0)-\pi\|\leq\|p(t)-\pi\|\leq e^{-\alpha t}\|p(0)-\pi\|,$$

where $-\alpha$ and $-\beta$ are the extreme (real) negative characteristic values of Q as an operator on the Hilbert space $l_2(\mu)$, and where $\|x\|^2\equiv\sum|x_n|^2\mu_n$. {The proof of (*) contains some misprints; in the last line of formulas each extreme member should be doubled, and π should be replaced by r; also the second "min" should be "max".}

D. G. Kendall (Oxford)

2311:

Good, I. J. Legendre polynomials and trinomial random walks. Proc. Cambridge Philos. Soc. **54** (1958), 39–42.

The note is concerned with one-dimensional walks in which there are positive probabilities of making steps in either direction, and also of not moving. The probability of arriving at the origin at the nth step is expressed in terms of Legendre polynomials, for which an asymptotic expansion is proposed.

S. K. Zaremba (Swansea)

2312:

Good, I. J. Random motion and analytic continued fractions. Proc. Cambridge Philos. Soc. **54** (1958), 43–47.

For the case of a one-dimensional random walk in which the probabilities of moves in either direction, and of no move, depend on the position of the particle, various probability generating functions are expressed in terms of continued fractions; the final formula for the probability of a return to the origin agrees with a result by T. E. Harris [Trans. Amer. Math. Soc. **73** (1952), 471–483; MR **14**, 567]. Ten particular cases are discussed in detail.

S. K. Zaremba (Swansea)

2313:

Lambert, François. Les problèmes d'attente. Cahiers Centre Études Rech. Oper. no. 2 (1959), 5–28.

A survey article of some aspects of queueing theory. The nature of the data, Monte-Carlo methods and analytic methods based on either servo-mechanisms or probability ideas are discussed. *D. V. Lindley* (Cambridge, England)

2314:

Roby, Norbert. Sur certains processus remarquables généralisant les processus de Poisson. C. R. Acad. Sci. Paris **248** (1959), 2945–2947.

Es seien X und Y zwei nicht-negative zufällige Variable und S_k die Summe von k unabhängigen und wie X verteilten zufälligen Variablen, wobei X nicht fast sicher verschwinde und Y von allen S_k unabhängig sei. Die Folge $\tau_k = Y + S_{k-1}$, $k = 1, 2, \cdots$ werde als eine Folge zufälliger Zeitpunkte interpretiert. Es bedeute $N(t, t')$ die Anzahl solcher Zeitpunkte im Intervall $[t, t'[$ und T_a die Wartezeit auf einen solchen Zeitpunkt vom Augenblick a an. Der Verf. drückt die Verteilungen von $N(t, t+s)$ und T_a mit Hilfe der Repartitionsfunktionen der S_k und der Erwartungen $E(t) = \mathscr{E}\{N(0, t)\}$ aus und untersucht ihr asymptotisches Verhalten für $t \to +\infty$. Im Fall $\mathscr{E}(X) = 1/\lambda < +\infty$ erhält man $\lim_{t\to\infty} E(t)/t = \lambda$, und hieraus resultieren im zeitlichen Mittel asymptotische Ausdrücke für die genannten Verteilungen, die nicht mehr von Y abhängen. Ist die Verteilung von Y gleich der Grenzverteilung von T_a, so gehen diese drei asymptotischen Formeln in für jedes t gültige Identitäten über, so daß insbesondere $N(t, t+s)$ nicht von t abhängt. Hat darüber hinaus X dieselbe Verteilung wie Y, so erhält man die Poissonschen Prozesse. Im Fall $\mathscr{E}(X) = +\infty$ gilt $\lim_{t\to\infty} E(t)/t = 0$, und $\mathscr{P}\{N(t, t+s) = n\}$ strebt im zeitlichen Mittel gegen 0 oder 1 je nachdem $n > 0$ oder $n = 0$ ist. Unter geeigneten Voraussetzungen hat man $E(t) \sim t/\log t$.

K. Krickeberg (Aarhus)

2315:

Sedrakyan, L. G. Two problems of the statistical strength theory. Akad. Nauk Armyan. SSR Dokl. **26** (1958), no. 3, 135–140. (Russian. Armenian summary)

By means of elementary combinatorial probability methods the following two problems are investigated: (1) To determine the strength distribution function of chains consisting of n links, if the strength distribution functions of the links are given; (2) If a cable is braided of n wires of equal length and cross-section, with their strength distribution function given, to determine the probability that under the influence of a given exterior load, $c \leqq n$ wires are breaking. The author points out that many questions occurring in the statistical strength theory reduce to the above two problems.

O. Onicescu (Bucharest)

STATISTICS

See also 2540.

2316:

★Lindgren, B. W.; and McElrath, G. W. Introduction to probability and statistics. The Macmillan Co., New York, 1959. xiii + 277 pp. $6.25.

This carefully written book on the essentials of elementary statistical theory is intended as a text for undergraduate students with a background in mathematics through integral calculus. The treatment is restricted to univariate distributions, for several results references are given instead of full proofs, and topics that are either obsolete or of little value to beginners are omitted. By these restrictions the authors are able to include introductions to some topics usually not mentioned in elementary books, such as the central limit theorem, the Kolmogorov-Smirnov and Wilcoxon-Mann-Whitney tests, sequential analysis, analysis of variance, and decision theory. There are about 300 exercises. Contents: Simple probability models; Random variables and distribution functions; Discrete distributions; Continuous distributions; Sums of random variables; Sampling. Presentation and description of data; Testing statistical hypotheses; Testing and estimating location; Estimating and testing variability; Comparison of two populations; Some further topics; Notation; Glossary of symbols; Tables; Some references for further study; Answers to problems; Index.

D. M. Sandelius (Göteborg)

2317:

★Croxton, Frederick E. Elementary statistics with applications in medicine and the biological sciences. Dover Publications, Inc., New York, 1959. vii + 376 pp. $1.95.

A clear, non-mathematical account of the elementary classical calculations. *M. Stone* (Cambridge, England)

2318:

James, G. S. On moments and cumulants of systems of statistics. Sankhyā **20** (1958), 1–30.

This paper presents a detailed study of k-statistics, i.e., functions of the sample values whose expectations are equal to the cumulants of the population. The results can be modified to obtain the sampling properties of a general set of statistics z_j, which are homogeneous polynomial symmetric functions of the sample values, of degree j, $j = 1, 2, \cdots$. Rules for finding expressions of the moments of the z_j's in terms of population moments or cumulants, and the cumulants of the z_j's in terms of population cumulants, are described. Multivariate extensions are also discussed. *I. Olkin* (East Lansing, Mich.)

2319:

David, F. N. The z-test and symmetrically distributed random variables. Biometrika **46** (1959), no. 1/2, 123–129.

For the usual one-way classification in the analysis of variance, the author investigates the stability of the α significance level when the underlying variables have a common symmetrical distribution. The method is an alternative to that of David and Johnson [Ann. Math. Statist. **22** (1951), 382–392; MR **13**, 143] and makes use of an expansion, in terms of cumulants, for $z=\frac{1}{2}\log F$, from which the effects of unequal group frequencies and non-normality can be studied.

I. Olkin (East Lansing, Mich.)

2320:

Kshirsagar, A. M. Bartlett decomposition and Wishart distribution. Ann. Math. Statist. **30** (1959), 239–241.

The Bartlett decomposition theorem gives the relation between the Wishart distribution and the joint distribution of independent normal and chi-square variables. A number of proofs of this theorem are available, and another proof is now given.

I. Olkin (East Lansing, Mich.)

2321:

Johler, J. Ralph; and Walters, Lillie C. Mean absolute value and standard deviation of the phase of a constant vector plus a Rayleigh-distributed vector. J. Res. Nat. Bur. Standards **62** (1959), 183–186.

Given a Rayleigh-distributed vector in two dimensions; denote its mean square length by k^2. To this vector add the vector of components (1, 0). (The vector field thus obtained is the same as the equilateral bivariate normal distribution viewed non-centrally.) The authors compute the first two absolute moments of the known distribution of the phase of this vector field. The results are reported to 8 significant digits, with checks to about 6 or 7, for 181 values of k^2 in the range 0.01 to 1000. To reduce the phase to a single-valued function for their integrations, the authors make a cut along the negative x-axis. In applications to detection apparatus, the counterpart of this cut might be a gate of some sort, which might not occur at exactly the location taken for the cut in the calculation. *A. Blake* (Ann Arbor, Mich.)

2322:

Rider, Paul R. Quasi-ranges of samples from an exponential population. Ann. Math. Statist. **30** (1959), 252–254.

The author derives the distribution and the cumulants of the difference between the $(n-r)$th and $(r+1)$th order statistics of a random sample of size n from an exponential distribution. *D. M. Sandelius* (Göteborg)

2323:

Ramasubban, T. A. The generalized mean differences of the binomial and Poisson distributions. Biometrika **46** (1959), no. 1/2, 223–229.

In Biometrika **45** (1958), 549–556 the author defined the generalized mean differences

$$\Delta_r = \sum_{ij} |i-j|^r Pr\{X=i\}Pr\{X=j\}$$

for a discrete p.d. Expressions for Δ_1, mostly in terms of hypergeometric functions, were obtained for a number of underlying probability distributions, e.g. (i) Binomial, (ii) Poisson, (iii) negative binomial. Expressions for Δ_r are now given for cases (i) and (ii).

I. Olkin (East Lansing, Mich.)

2324:

Saw, J. G. Estimation of the normal population parameters given a singly censored sample. Biometrika **46** (1959), no. 1/2, 150–159.

2325:

★Kullback, Solomon. Information theory and statistics. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1959. xvii+395 pp. $12.50.

This book is about statistics; in particular, the testing of statistical hypotheses. Since statistics has to do with gathering and using information (in the wide sense), and since "information theory" (in the Shannon sense) purports to be a quantitative theory of information and its handling, one is tempted to suppose that statistics could be given an information theoretic basis. This book represents a more or less unified account of the author's pioneering efforts in this direction. Starting with a few fairly simple postulates of an information theoretic nature, the author derives an impressive array of old and new results in statistical hypothesis testing and related areas. The basic ideas are as follows. (1) The "directed divergence" from one simple hypothesis to another. Under H_1 (H_2) let probability have distribution μ_1 (μ_2) on the measurable space $[\mathcal{X}, \mathcal{F}]$, where μ_1 and μ_2 have the same null sets. Then $I(1:2;\ \mathcal{X})=\int_{\mathcal{X}} d\mu_1 \log d\mu_1/d\mu_2$ is the average usefulness (reviewer's term) of a sample $x\in\mathcal{X}$ for discriminating against H_2 when H_1 is the fact. For instance, $I(1:2;\ \mathcal{X})\geq 0$, with equality if and only if $\mu_1=\mu_2$ on $\mathcal{F}$. Also, if $T:\mathcal{X}\to\mathcal{Y}$ is a statistic with induced distributions $\nu_i=\mu_i T^{-1}$, $i=1,\ 2$, then $I(1:2;\ \mathcal{X})\geq I(1:2;\ \mathcal{Y})=\int_{\mathcal{Y}} d\nu_1 \log d\nu_1/d\nu_2$, with equality if and only if T is sufficient for discriminating between H_1 and H_2. (2) The directed divergence from a sample to a hypothesis. Suppose an observation yields the value $\hat{\theta}$ for a real valued statistic $T(x)$. The "minimum discrimination information statistic $\hat{I}(*:\ H_2)$" between the sample $T(x)=\hat{\theta}$ and a simple hypothesis H_2 is defined as $\hat{I}(*:\ H_2)=\inf_1 I(1:2;\ \mathcal{X})$, where the infimum is over all probability measures μ_1 which have the property $\int_{\mathcal{X}} d\mu_1 T(x)=\hat{\theta}$. (Let $I(1:2;\ \mathcal{X})=+\infty$ unless $\mu_1\ll\mu_2$.) If a minimizing μ_1 exists then it is a member of an exponential family generated by T and μ_2, viz.,

$$d\mu_1/d\mu_2 = [M_2(\hat{\tau})]^{-1}\exp[\hat{\tau}T(x)],$$

where $M_2(\tau)=\int_{\mathcal{X}} d\mu_2 \exp[\tau T(x)]$, and where $\hat{\tau}$ is chosen so that $M_2'(\hat{\tau})/M_2(\hat{\tau})=\hat{\theta}$; the value of $\hat{I}(*:H_2)$ is then $\hat{I}(*:H_2)=\hat{\tau}\hat{\theta}-\log M_2(\hat{\tau})$. Define $\hat{I}(*:H_2)=0$ if no such $\hat{\tau}$ exists. For composite H_2 define $\hat{I}(*:H_2)$ as the infimum over simple members of H_2 of $\hat{I}$ defined above. The extension to vector valued statistics is obvious. (3) Uses of $\hat{I}$. The statistic $\hat{I}(*:H)$ is to be regarded as the information theoretic amount by which hypothesis H is inconsistent with the sample $T(x)=\hat{\theta}$. In a classification problem, for instance, if $H_1, \cdots, H_r$ are mutually exclusive and exhaustive hypotheses that the sample $T(x)=\hat{\theta}$ was drawn from population (or population class) $1, \cdots, r$, then the sample is to be assigned to the population class

j for which $\hat{I}(*:H_j)$ is smallest. (This works out to be a maximum likelihood procedure when the probability measures under (simple) $H_1, \cdots, H_r$ are members of an exponential family generated by $T(x)$.) In another application, the critical region for rejecting a null hypothesis H_2 in favor of an alternative H_1 is determined by $\hat{I}(*:H_2)-\hat{I}(*:H_1)\geqq c$, where c is determined by the size of the test. (When the hypotheses are simple, use of the statistic $T(x)=\log d\mu_1/d\mu_2$ gives the Neyman-Pearson most powerful critical region.) Using these methods, the author gives (in chapters 6–13, comprising 244 pages) detailed and often numerical discussion of examples in multinomial and Poisson populations, contingency tables, and multivariate normal populations. Substantial problem lists appear at the end of each chapter. Referencing is extremely thorough in all sections, including the problem sections. {In criticism, the reviewer remarks that the items designated (1)–(3), above, first appear in chapters 1, 3, and 5, respectively, so that the reader is exposed to much secondary material before he reaches the author's main thesis; the definition of critical region on p. 85 is not clear; in several places, the discussion of tests based on the symmetrized $J(1:2)=I(1:2)+I(2:1)$ could well have been omitted. These are minor faults, however.} The author states in the preface, "Applications to more general stochastic processes, including sequential analysis, will make a natural sequel, but are outside the scope of this book." *S. P. Lloyd* (Murray Hill, N.J.)

2326:

Gnanadesikan, R. Equality of more than two variances and of more than two dispersion matrices against certain alternatives. Ann. Math. Statist. **30** (1959), 177–184.

Using the union-intersection principle [Roy, same Ann. **24** (1953), 220–238; MR **15**, 241] the author proposes two simultaneous tests and corresponding confidence regions. The first test is that of the equality of the variances of $k+1$ (>2) univariate normal distributions, against the alternative that at least one of the k last variances is different from the first variance. The acceptance region of the joint test is the intersection of the acceptance regions for k variance ratio tests. To determine these tests the author imposes on them the conditions of equal size and local unbiasedness. The second test is an extension of the first and deals with the hypothesis of equal dispersion matrices, one of which is chosen as a standard. The properties of the power are investigated for the first test. Due to lack of tables the tests are not ready for immediate application.

D. M. Sandelius (Göteborg)

2327:

Murteira, B. On type A regions for Pólya distributions. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 327–330.

The author gives a direct proof of the theorem of Karlin [Ann. Math. Statist. **28** (1957), 281–308; MR **19**, 475] that if a testing problem is of strictly Pólya type 3, the type A critical regions consist of a union of at most two semi-infinite intervals. *H. Rubin* (Eugene, Ore.)

2328:

Good, I. J. Significance tests in parallel and in series. J. Amer. Statist. Assoc. **53** (1958), 799–813.

The author is mainly concerned with significance tests "in parallel", i.e. several tests applied to the same body of data, especially either (a) tests against several different hypotheses or (b) several inefficient tests against a fixed alternative. He suggests that if the tail-area probabilities of the test statistics are $P_1, P_2, \cdots, P_n$, then, if the statistician can think of nothing better to do, he may reasonably summarize them by their harmonic mean, $n/\sum P_i^{-1}$, or possibly by a weighted harmonic mean, $\sum w_i/\sum w_iP_i^{-1}$. This rule is justified by a "neo-Bayesian" argument, and this, together with material in the appendices, is of interest in connection with the relation between orthodox tail-area significance tests and Bayesian inference, as previously discussed by the author, by H. Jeffereys [*Theory of probability*, Oxford, 1939, 1948, chap. 7; MR **1**, 151] and by D. V. Lindley [Biometrika **44** (1957), 187–192]. Tests in parallel are contrasted with tests "in series", one test to each of several bodies of data, for which a well-known suggestion for combination of tail-area probabilities was made by R. A. Fisher [*Statistical methods for research workers*, Oliver and Boyd, Edinburgh, 1925; § 21.1].

F. J. Anscombe (Princeton, N.J.)

2329:

Turner, Malcolm E.; and Stevens, Charles D. The regression analysis of causal paths. Biometrics **15** (1959), 236–258.

This is essentially an explanation of Sewall Wright's "path coefficients", but using ordinary regression coefficients instead of standardized ones. The underlying assumptions are explicitly stated. Essentially we have certain measurable variables x_a, acting as "original causes", and hypothetical variables η_s which are usually assumed to be linear functions of certain specified x_a and other η_t. The non-zero coefficients $\alpha_{sa}=\partial\eta_s/\partial x_a$ and $\alpha_{st}=\partial\eta_s/\partial\eta_t$ are the "path coefficients" to be estimated. However, instead of the η_s we observe $y_s=\eta_s+\varepsilon_s$, where ε_s is a random error term. By considering the regressions of the y_s on each other and on the x_a it is shown how we can in many cases estimate the path coefficients, and find confidence intervals. A simple example is included in which there is "feedback", i.e., in which there are two variables η_1 and η_2 which reciprocally influence each other. *C. A. B. Smith* (London)

2330:

Watanabe, Yoshikatsu; Yamamoto, Tadayuki; Satô, Takeshige; Fujimoto, Takaaki; Inoue, Mamoru; Suzuki, Takesi; and Uno, Tomoko. Some contributions to order statistics. (Continued) J. Gakugei Tokushima Univ. Math. **9** (1958), 31–86.

This paper, which continues an earlier paper by Watanabe et al. [same J. **8** (1957), 41–90; MR **20** #5545] gives detailed formulae by means of which higher moments of order statistics of samples from a normal distribution can be computed. Tables of all third and fourth moments, mixed and univariate, about the origin are given, with 7 decimals, for samples of sizes 2(1)5.

D. M. Sandelius (Göteborg)

2331:

Dunn, Olive Jean. Estimation of the medians for dependent variables. Ann. Math. Statist. **30** (1959), 192–197.

Let, for $i=1, 2$, y_i be a random variable having a continuous distribution with median ν_i, let $z_{i1} \leqq \cdots \leqq z_{in}$ be the ordered observations on y_i in a random sample of size n from the joint distribution of y_1 and y_2, and let E_i be the event that $z_{ir} < \nu_i < z_{i,r+s+1}$, where r and s are positive integers such that $2r+s=n$. The author proves that

$$\text{Prob}\,(E_1E_2) \geqq \text{Prob}\,(E_1)\,\text{Prob}\,(E_2).$$

D. M. Sandelius (Göteborg)

2332:
Kitabatake, Satoshi. A remark on a non-parametric test. Math. Japon. **5** (1958/59), 45–49.

Let $F_0(x)$ be a continuous distribution function, with inverse $F_0^{-1}(y)$, and let a_i be such that $F_0(a_i)-F_0(a_{i-1})=1/n$ $(i=1, \cdots, n)$ with $a_0=-\infty$, and $a_n=+\infty$. Now consider a sample $x_1, \cdots, x_N$ from an absolutely continuous distribution function $F_1(x)$. Let v be the number of the intervals (a_i, a_{i-1}) $(i=1, \cdots, n)$ which contain no values of the sample.

Let $G(y)=F_1(F_0^{-1}(y))$ and $g(y)=dG(y)/dy$. The author shows that as N, $n\to\infty$ with $N/n=r$, a constant, the random variable v/n is asymptotically normal with mean

$$(1) \qquad \int_0^1 e^{-rg(y)}dy,$$

and variance

$$(2) \quad n^{-1}\left[\int_0^1 (e^{-rg(y)}-e^{-2rg(y)})dy - r\left(\int_0^1 g(y)e^{-rg(y)}dy\right)^2\right].$$

If $F_1(x)$ differs from $F_0(x)$ then

$$\int_0^1 e^{-rg(y)}dy > e^{-r}.$$

This together with the fact that the variance given by (2) tends to 0 as $n\to\infty$ means that v/n is a consistent test for the hypothesis that the sample comes from $F_0(x)$ against the hypothesis that it comes from any absolutely continuous distribution $F_1(x)$ which differs from $F_0(x)$.

S. S. Wilks (Princeton, N.J.)

2333:
Naddeo, Alighiero. Analisi della variabilità delle relazioni statistiche. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 621–630.

2334:
Sarmanov, O. V. The maximum correlation coefficient (symmetric case). Dokl. Akad. Nauk SSSR **120** (1958), 715–718. (Russian)

Let $F(x, y)$ be a symmetric probability density defining the correlation between the random variables x and y in the domain $(a \leqq x, y \leqq b)$ and $p(x)=\int_a^b F(x, y)dy$ and let us suppose that $K(x, y) = F(x, y)/(p(x)p(y))^{1/2}$ is integrable with respect to x and y. Suppose now that $\lambda_0, \lambda_1, \lambda_2, \cdots$, are the spectral values of $K(x, y)$. If $r_0(x)$ is any function with dispersion for which $\int_a^b r_0(x)p(x)dx=0$, and putting

$$r_k(x) = \int_a^b r_{k-1}(y)\,F(x, y)/p(x)\,dy, \quad k = 1, 2, \cdots,$$

then, for sufficiently large k, $\lambda_1^{-1} \approx r_k(x)/r_{k-1}(x)$. [See the author's paper, Dokl. Akad. Nauk SSSR **53** (1946), 773–776; MR **8**, 467.]

By definition $R^*=\lambda_1^{-1}$ is the maximal coefficient of correlation corresponding to $F(x, y)$. It is shown that two random variables x and y are independent, if and only if $R^*=0$. If the correlation is rectilinear and R is the usual coefficient of correlation between x and y, then $R^*=R$. The discrete case is treated analogously.

O. Onicescu (Bucharest)

2335:
Fréchet, Maurice. A note on simple correlation. Math. Mag. **32** (1958/59), 265–268.

Remarks on properties to be sought in a measure of correlation, and on the extent to which various measures possess them.

2336:
Smith, C. D. Some further notes on the theory of correlation. Math. Mag. **32** (1958/59), 269–270.

Remarks on interpretation of data in terms of regression.

2337:
Bross, Irwin D. J. Note on an application of the Schumann-Bradley table. Ann. Math. Statist. **30** (1959), 581–583.

Schumann and Bradley [same Ann. **28** (1957), 902–920] developed a procedure and provided a table for comparing the sensitivity of "similar" experiments in a "fixed effects" analysis of variance model. This paper points out that for "identical" experiments, i.e., the experiments can be viewed as independent replicates, the same table can be used as a test of the "random effects" model. An approximation is suggested for the general case.

C. J. Maloney (Frederick, Md.)

2338:
Kshirsagar, A. M. Distribution of the "blocks adjusted for treatments" sum of squares in incomplete block designs. Ann. Math. Statist. **30** (1959), 246–249.

For a model with normally and independently distributed block effects and errors, the distribution referred to in the title is found to be that of a sum of weighted chi-squares, the weights being equal if the design obtained by taking blocks as treatments and treatments as blocks in a balanced incomplete block design.

D. M. Sandelius (Göteborg)

2339:
Roy, J. A class of two replicate incomplete block designs. Biometrics **15** (1959), 259–269.

Experiment designs having two replicates are given together with the intra- and inter-block analysis. These designs are obtained by (i) taking a two associate partially balanced incomplete block design; (ii) forming one block of size two with each pair of first associate treatments; and (iii) dualizing the resulting design.

M. Zelen (Washington, D.C.)

2340:
Giri, N. C. On row-balance in P. B. I. B. designs. J. Indian Soc. Agric. Statist. **9** (1957), 168–178.

The author presents two families of partially balanced incomplete block designs with two associate classes which can be used for two-way elimination of heterogeneity. A

characteristic of these experiment designs is that treatments do not have to appear a constant number of times within each row. Both the intra- and inter-block analyses are given. *M. Zelen* (Washington, D.C.)

2341:
Robson, D. S. A simple method for constructing orthogonal polynomials when the independent variable is unequally spaced. Biometrics **15** (1959), 187–191.

A recursive procedure is given for the construction of orthogonal polynomials when the values of the independent variable are unequally spaced. The construction of the polynomials for use in a 3×4 factorial design is illustrated. *M. Muller* (New York, N.Y.)

2342:
Biggers, J. D. The estimation of missing and mixed-up observations in several experimental designs. Biometrika **46** (1959), no. 1/2, 91–105.

The straightforward application of regression theory to the analysis of balanced experimental designs with one or more missing observations may be arithmetically complex. A method is given for exploiting the symmetry of such designs which enables one to write down at once a set of simultaneous equations for the least squares estimates of the missing values. The method is extended to cover the case of mixed-up observations (for which the total of a group of observations, but not their individual values, is known).

The procedure is applied to a number of experimental designs including randomized block, cross-over, latin square, and split plot. *P. Meier* (Chicago, Ill.)

2343:
Shah, B. V. On a generalisation of the Kronecker product designs. Ann. Math. Statist. **30** (1959), 48–54.

A method of constructing partially balanced incomplete block designs for $v=pq$ treatments with three associate classes is given. The treatments are written in p rows and q columns and the association scheme is such that any pair of treatments in the same row is a first associate pair, any pair of treatments in the same column is a second associate pair while any other pair of treatments is a third associate pair. The method is a generalization of the technique used by Vartak [same Ann. **26** (1955), 420–438; MR **17**, 227] and Sillitto [Biometrika **44** (1957), 278–279]. {The reviewer noticed two mistakes: in theorem 3.1 read r_p^* for γ_p^*; on page 52, line 8 read $r_1^*r_2^*$ for $r_2^*r_2^*$.} *S. S. Shrikhande* (Chapel Hill, N.C.)

2344:
Steinhaus, H.; and Zubrzycki, S. On comparing two production processes and on the principle of dualism. Zastos. Mat. **3** (1958), 229–257. (Polish. Russian and English summaries)

A Polish version of "On the comparison of two production processes and the role of dualism", Colloq. Math. **5** (1957), 103–115 [MR **19**, 1205]. *S. K. Zaremba* (Swansea)

2345:
Kordonskiĭ, H. B. On a distribution for the number of spoiled articles in shipments. Teor. Veroyatnost. i. Primenen. **3** (1958), 354–358. (Russian. English summary)

2346:
Firescu, D. Fonctions d'estimation efficientes pour les probabilités de passage d'une chaîne de Markoff. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **7** (1958), no. 20, 37–47. (Romanian. Russian and French summaries)

The author investigates the efficiency of the estimation functions n_{ij}/n_i of the probabilities p_{ij} of a Markoff chain, n_i and n_{ij} being the respective numbers of the apparition of the event a_i or of the succession of events a_i, a_j in n experiences. The asymptotical laws of the vectors

$$\|n_{i_1j_1}n_{i_2j_2}\cdots n_{i_rj_r}\| \quad\text{and}\quad \left\|\frac{n_{i_1j_1}}{n_{i_1}}\frac{n_{i_2j_2}}{n_{i_2}}\cdots\frac{n_{i_rj_r}}{n_{i_r}}\right\|,$$

conveniently normed, are Gauss's laws. Consequently, the individual estimation functions $n_{i_1j_1}/n_{i_1}$, $n_{i_2j_2}/n_{i_2}$, $\cdots$, $n_{i_rj_r}/n_{i_r}$ are asymptotical Gaussian. *O. Onicescu* (Bucharest)

2347:
Guest, P. G. Methods for numerical calculations with the type I counter. Austral. J. Phys. **11** (1958), 143–153.

Schemes are proposed for the numerical evaluation of the well-known formulae giving the expected number of counts in a given time, and its derivative, when events describable as a Poisson process are counted by a Type I counter (i.e. by one in which every count is followed by a period of insensitivity of fixed duration), starting at a moment when the counter is sensitive. Exact results are compared with those obtained from the obvious asymptotic formulae. The variance of the actual number of counts, as well as the effect of placing two counters in series, are also discussed. *S. K. Zaremba* (Swansea)

NUMERICAL METHODS

See also 1977, 1980, 2128, 2347, 2376, 2453, 2454, 2535.

2348:
Pasta, John R.; and Ulam, S. Heuristic numerical work in some problems of hydrodynamics. Math. Tables Aids Comput. **13** (1959), 1–12.

Some numerical approaches to the solution of problems dealing with the dynamic behavior of continua are reviewed. The selection of a replacement for a continuum, and time, by a set of "generalized" points in the appropriate coordinate system is considered. The paper concludes by reviewing the results of some numerical computations concerning the idealized motion of: (1) a heavy fluid on top of a lighter one and (2) a heavy gas on top of a light gas. *M. Muller* (New York, N.Y.)

2349:
★**Fox, Leslie. Minimax methods in table construction.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 233–244. Edited by R. E. Langer. Publication no. 1 of the Mathematics

Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

This gives a general survey of the modern methods of interpolation available for meeting the needs of a user of a mathematical table who has to compute values with the aid of a desk machine. The emphasis is on high convergence for the methods used, with a description of the auxiliary quantities needed in the table in order to carry out the interpolation. A fuller account is given in L. Fox, *The use and construction of mathematical tables* [Nat. Phys. Lab. Math. Tables, Vol. 1. H. M. Stationery Office, London, 1956]. *J. C. P. Miller* (Cambridge, England)

2350:

Bodewig, E. Zum Matrizenkalkul. III. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 305–312.

[For parts I, II, IV, V, see same Proc. **58** (1955), 95–106, 260; **59** (1956), 301–304; **60** (1957), 82–87, 242–247; MR **17**, 339; **19**, 936, 1153, 769.] Let the square matrix A be written in the form $I-L-U$, where I=unit matrix, U=upper, L=lower triangular matrix (main diagonal excluded), and let R_1 be an approximate inverse of A. The author proposes to calculate A^{-1} by the formula

$$A^{-1} = R_1(I+C)(I+C^2)(I+C^4)\cdots,$$

where $C=I-AR_1$. Some methods for obtaining R_1 are indicated. A special matrix calculus appropriate to "band" matrices ($a_{ik}=0$ for $|i-k|\leq p$ for some $p<n$), using parallels to the main diagonal rather than rows and columns, is also developed in the paper.

P. Henrici (Los Angeles, Calif.)

2351:

Causey, Robert L. Computing eigenvalues of non-Hermitian matrices by methods of Jacobi type. J. Soc. Indust. Appl. Math. **6** (1958), 172–181.

The famous Jacobi diagonalization algorithm for symmetric matrices uses 2-dimensional rotations in planes spanned by two coordinate axis. By a sequence of such rotations the off-diagonal elements of the given matrix are diminished. Recently H. H. Goldstine [H. H. Goldstine and L. P. Horwitz, J. Assoc. Comput. Mach. **6** (1959), 176–195; MR **21** #426] has shown that a generalized Jacobi method using 2-dimensional unitary rotations may be established for diagonalizing normal matrices. Furthermore a theorem of Schur states that an arbitrary matrix is unitarily similar to a triangular matrix. Greenstadt as well as Lotkin proposed algorithms using 2-dimensional unitary rotations for transforming a matrix into a triangular one.

The author reviews, examines and criticizes both methods. He lists test matrices where neither method converges. Therefore some modification of the methods is needed if they are to triangularize all matrices. Nevertheless they probably triangularize the majority of matrices which occur in practical problems. The author also carried out numerical experiments with Greenstadt's method improving the computational technique. He concludes that the method involves a prohibitive amount of computing and converges at best linearly. It seems therefore that either Jacobi's idea is not appropriate for computing eigenvalues of general matrices or that the corresponding theory is not yet sufficiently developed.

E. Stiefel (Zürich)

2352:

Gregory, Robert T. Results using Lanczos' method for finding eigenvalues of arbitrary matrices. J. Soc. Indust. Appl. Math. **6** (1958), 182–188.

In 1950 C. Lanczos [J. Res. Nat. Bur. Standards **45** (1950), 255–282; MR **13**, 163] proposed the following method for computing the eigenvalues of a symmetric matrix A. Beginning with an arbitrary start vector, the vectors iterated by A are orthogonalized. Because of the symmetry of A the ordinary orthogonalization algorithm is simplified and furnishes a three term recursion formula for computing the successive orthogonal vectors. Later on Lanczos generalized his procedure to non-symmetric matrices replacing orthogonalization by bi-orthogonalization with respect to A and its transposed matrix.

In the paper experiences with Lanczos' method are described. In order to keep rounding-off errors small, the method is improved introducing the following modifications. The bi-orthogonalization is carried out without taking into account the simplification mentioned above. This is to say that each new iterated vector is exactly made orthogonal to the previous vectors. Moreover double-precision and scaling are used. An example proves that the modified method is more stable than the original one.

By Lanczos' algorithm the matrix A is transformed into a tri-diagonal matrix B (Jacobi type). The author computes the eigenvalues of B by determinant evaluation. The reviewer would recommend Rutishauser's QD-algorithm for this purpose. *E. Stiefel* (Zürich)

2353:

Weisfeld, Morris. A remark on a paper of R. T. Gregory. J. Soc. Indust. Appl. Math. **7** (1959), 131.

In his code of Lanczos' method R. T. Gregory [see preceding review] used a scaling procedure in order to reduce the accumulation of rounding-off errors. The author discusses the influence of this device on the computation of the eigenvectors of a given matrix. He shows that whatever accuracy is gained in scaling for finding eigenvalues is lost by the additional multiplications needed in computing the eigenvectors.

E. Stiefel (Zürich)

2354:

Frank, Werner L. Computing eigenvalues of complex matrices by determinant evaluation and by methods of Danilewski and Wielandt. J. Soc. Indust. Appl. Math. **6** (1958), 378–392.

The author investigates the calculation of eigenvalues of matrices by applying the so-called Muller method [Math. Tables Aids Comput. **10** (1956), 208–215; MR 18, 766] to the characteristic polynomial. The latter is computed (a) by Danielewski's method and (b) for every λ occurring in Muller's method by evaluating $\det(A-\lambda I)$ directly with an elimination procedure. In applying the methods to two test matrices, the author observes a severe loss of accuracy (in one case for method (a), in a second test even for (b)). The failure of method (a) was to be expected, since the roots of a polynomial of high order are

known to be extremely sensitive to changes of the coefficients. The failure of method (b) is not so obvious since the eigenvalues are well spaced. As the author indicates, the method of Wielandt is superior to the methods mentioned above. *H. Rutishauser* (Zürich)

2355:

Bruins, E. M. On fourth and fifth roots. Bull. Coll. Arts Sci. Baghdad **2** (1957), 99–103.

The author shows how the method of Heron [Metrica I, 8] for the computation of $\sqrt{Q^2}$ (where Q^2, not Q, is a known number) may be extended to the computation of fourth and fifth roots, by a computation of only one square root, e.g., from a first approximation $\sqrt[5]{P^5} \sim a$ (which may be rather crude) the author obtains as a much better approximation

$$p_1 = \frac{(5P^{10}+20a^5P^5)^{1/2}-P^5+a^5}{(5P^{10}+20a^5P^5)^{1/2}-P^5+6a^5}\cdot 2a.$$

For $P^5=2$, $a=1$ he obtains $p_1=1.1486439$, whereas $\sqrt[5]{2}=1.14869837$. *S. C. van Veen* (Delft)

2356:

Gross, O.; and Johnson, S. M. Sequential minimax search for a zero of a convex function. Math. Tables Aids Comput. **13** (1959), 44–51.

Given $f(x)$ convex and continuous on $[a, b]$, $f(a)>0$, $f(b)<0$. In N steps (involving N evaluations of f) one seeks to find the smallest interval in which a zero of $f(x)$ certainly lies, no matter what $f(x)$ is. At each step, one uses the best procedure for the worst possible situation; hence the phrase "sequential minimax search". The authors present an algorithm, derived from the theory of dynamic programming, for solving this problem. Applied to several examples, the algorithm seems to give good results in comparison with some other methods, such as bisection, which use no more information.

M. A. Hyman (Yorktown Heights, N.Y.)

2357:

Rao, C. V. Govinda. A solution of the non-linear differential equation of the rotating machine oscillator. J. Franklin Inst. **265** (1958), 29–42.

An analysis of a d-c series generator—separately excited motor oscillator gives rise to the equation

$$(*) \qquad Lq''+(R-\alpha)q'-(\beta/3)q'^3+(\gamma/5)q'^5+K\phi q = 0.$$

Leinard's graphical method is used to determine the existence of periodic solution of (*) and the effect of system parameters on stability of the oscillator.

R. T. Herbst (Winston-Salem, N.C.)

2358:

Kotova, L. N. A linearized estimate of the error involved in the numerical integration of the simultaneous differential equations of the fundamental problem of external ballistics. Dokl. Akad. Nauk SSSR **121** (1958), 418–421. (Russian)

The author discusses the estimation of the errors in a numerical solution of the differential equations of exterior ballistics by the Adams method.

D. G. Aronson (Minneapolis, Minn.)

2359:

Squire, William. Approximate solution of linear second order differential equations. J. Roy. Aero. Soc. **63** (1959), 368–369.

2360:

Douglas, A. S. On the numerical solution of a class of partial differential equations. Proc. Cambridge Philos. Soc. **54** (1958), 214–218.

It is desired to find the solution P_0 of Schrödinger's equation

$$\nabla^2 P+2(E-V)P = 0$$

with homogeneous boundary conditions corresponding to the smallest eigenvalue E_0. Hartree (unpublished note) has shown how to obtain P_0 from asymptotic ($t\to\infty$) solutions of

$$\nabla^2 Y+2(E^*-V)Y = \frac{\partial Y}{\partial t},$$

where E^* is a judiciously chosen constant. The author attacks here the one-dimensional equation

$$\frac{\partial^2 W}{\partial r^2}+g(r)W = \frac{\partial W}{\partial t}$$

and presents several finite-difference schemes for obtaining its asymptotic solutions, some explicit and some implicit. He presents calculations performed on EDSAC I by himself and by J. Phelps illustrating the accuracy (compared to known solutions of Schrödinger's equation) and relative efficiency of various numerical schemes. Particularly effective is an implicit scheme where the time-increment is varied so as to strike hard in turn each of the higher harmonics in $W(r, 0)$, corresponding to eigenvalues $>E_0$. On page 214, line 21, read "non-negative"; on page 217, Table I, read "$a=\frac{1}{2}$, $b=1$, eigenvalues".

M. A. Hyman (Yorktown Heights, N.Y.)

2361:

★Collatz, Lothar. Approximation in partial differential equations. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 413–422. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

Given the N-dimensional region D with boundary Γ. One desires to solve the differential equation $E[u]=0$ in D with boundary conditions $F[u]=0$ on Γ. In most of the paper, E is a linear or quasi-linear second order differential operator; sometimes E is elliptic, sometimes parabolic, sometimes of mixed type (Tricomi's equation is considered). Suppose u satisfies the equation and the boundary conditions, while v satisfies only the equation: $E[v]=0$. In various cases the author and others have bounded $\max_D |u-v|$ with the maximum of a function involving the boundary values of u, v. One endeavors to choose v judiciously so that the latter maximum is small. For example, if E, F are linear operators and $E[v_i]=0$ ($i=1, 2, \cdots, p$) then $v=\sum_i^p a_i v_i$ where the a_i are chosen so that $\max_\Gamma \sum a_i F[v_i]$ is a minimum (Tchebycheff approximation). The author gives examples of the basic technique applied in various situations, and indicates how it is related to functional approximation in Banach and

pseudo-metric spaces. There is a bibliography of 21 items.

M. A. Hyman (Yorktown Heights, N.Y.)

2362:

Byhovskiĭ, È. B. The local solution with finite differences of Cauchy's problem for the equations of compressible fluid in the case of smooth initial values. Vestnik Leningrad. Univ. **13** (1958), no. 19, 39–44. (Russian. English summary)

The author studies the Cauchy problem for Euler equations with initial data which are zero outside a bounded set D, except that $\rho(x, 0)$ has the property: $\rho(x, 0)=\alpha>0$ outside D and $\rho(x, 0)\geqq\alpha$ on D. After multiplying the continuity equation by p_ρ/ρ, he puts the system in the form

$$(*) \qquad A\tilde{u}_t = B\tilde{u}+C\tilde{u}+f, \quad \tilde{u}(x, 0)\equiv 0,$$

where $\tilde{u}(x, t)=u(x, t)-u(x, 0)$, $u=(u^1, u^2, u^3, \rho, S)$, $x=(x_1, x_2, x_3)$. A is a diagonal matrix with positive entries, B is a symmetric first order differentiation matrix, C is a matrix and f is a vector involving only the initial data. A finite difference scheme for (*) is obtained by redefining all the entries in (*) on a lattice with mesh h in space directions and mesh $\Delta t=kh^2$ in time direction with suitable k, and replacing space derivatives by central difference quotients, and the time derivatives by forward difference quotients. It is shown that if the initial data are in C^6 and $p_\rho \leftarrow C^3$, then for $t\leqq T$ with sufficiently small T and for all x, the solution of the finite difference problem tends to the solution of (*) as $h\to 0$.

I. I. Kolodner (Albuquerque, N.M.)

2363:

Berger, J. M.; and Lasher, G. J. The use of discrete Green's functions in the numerical solution of Poisson's equation. Illinois J. Math. **2** (1958), 593–607.

The solution of an elliptic partial differential equation $E[u]=\rho$ with boundary conditions by finite difference methods involves the inversion of a large but sparse, well-structured matrix. This inversion is usually done iteratively or directly. The authors stress a third technique, based on a strict analogy between differential and difference systems; they construct the Green's function G for the difference equation and boundary conditions, then express the solution u as a sum of G times ρ. Computation of u by this technique appears to involve less work and smaller computation error than direct inversion; it also appears superior to iterative techniques in some cases, in particular if the solution u is desired only at a few points or if ρ is to be changed while retaining the boundary conditions. The authors give a careful discussion of the technique for Poisson's equation over a rectangle (the case of zero eigenvalues is discussed), including a detailed treatment of the computation error. Some of the methods and results parallel those obtained by the reviewer [Appl. Sci. Res. B **2** (1952), 325–351; MR **13**, 993], who combined Green's function methods (in a different form) with a technique based on regarding the difference equations as a recursion relation.

M. A. Hyman (Yorktown Heights, N.Y.)

2364:

★Szabó, J. Über die Auflösung der Poissonschen, biharmonischen und anderer ähnlich gebauter Differenzengleichungen mit Hilfe der Matrizenrechnung. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 9–23. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

The method is based upon the following observation: Consider the equation $\sum c_{\alpha\beta}A^\alpha F B^\beta = P$, where the matrix F is required, and the scalars $c_{\alpha\beta}$ and the other matrices are known. It is not required that F be square. If the decompositions $A=Q\Lambda Q^{-1}$ and $B=RMR^{-1}$ are known, with Λ and M diagonal, then the equation reduces to $\sum c_{\alpha\beta}\Lambda^\alpha(Q^{-1}FR)M^\beta=Q^{-1}PR$, and $Q^{-1}FR$ can be obtained, element by element.

In the case under consideration, F is the matrix of values of the dependent variable, and the matrices A and B are the familiar ones with the typical row

$$\cdots -1 \quad 2 \quad -1\cdots.$$

A. S. Householder (Oak Ridge, Tenn.)

2365:

★Siddiqi, Omar Ali. On the solution of Fredholm integral equations of second kind. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 341–344. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

The integral equation

$$\phi(x) = f(x)+\lambda\int_0^h K(x, y)\phi(y)dy$$

is approximately solved by $\phi=\sum_{K=0}^n c_K x^n$ and by determining the coefficients c_K such that the residual is of order x^{n+1} for small $|x|$. Two examples are given; in both of them $K(x, y)$ is a second order polynomial in x and y.

H. Bückner (Madison, Wis.)

2366:

★Чистова, Э. А. Таблицы функций Бесселя от действительного аргумента и интегралов от них. [Čistova, È. A. Tables of Bessel's functions with real argument and their integrals.] Vyčislitel'nyĭ Centr, Matematičeskie Tablicy. Izdat. Akad. Nauk SSSR, Moscow, 1958. 524 pp. (1 insert) 45 rubles.

The main tables give $J_r(x)$, $Y_r(x)$, $Ji_r(x)=\int_x^\infty t^{-1}J_r(t)dt$, $Yi_r(x)=\int_x^\infty t^{-1}Y_r(t)dt$ to 7S for $x=0(.001)15(.01)100$, $r=0, 1$. Auxiliary functions are tabulated for the range $x=0(.001).15$. Elsewhere (double) mean second differences $\tilde{\Delta}^2 f_0=\Delta^2 f_{-1}+\Delta^2 f_0$ are given for use in a Bessel formula $f_p=f_0+p\Delta f_0-\frac{1}{4}t(1-t)\tilde{\Delta}^2 f_0$ where these are necessary ($\tilde{\Delta}^2 f\geqq 16$). The insert gives $\frac{1}{4}t(1-t)$ to 5D for $t=0(.01).5$.

The values of J_0, J_1 were taken from the Harvard volumes and their integrals were computed by Simpson's rule partly using electronic equipment. The Y_r and Yi_r were computed on the computer BESM, using the representations by power series and by asymptotic series. Difference checks were applied. It is stated that the errors in the main table do not exceed .6 unit in the last place.

The introduction includes some graphs and summarizes the properties of the functions and gives asymptotic formulas from which values of the functions $x>100$ can be obtained.

The printing and format are satisfactory. (There is one flaw on p. 443.)

John Todd (Pasadena, Calif.)

2367:
★**Кармазина Л. Н.; и Чистова, Э. А. Таблицы функций Бесселя от мнимого аргумента и интегралов от них. [Karmazina, L. N.; and Čistova, Ê. A. Tables of Bessel's functions with imaginary argument and their integrals.]** Vyčislitel'nyĭ Centr, Matematičeskie Tablicy. Izdat. Akad. Nauk SSSR, Moscow, 1958. 329 pp. 37.15 rubles.

The main tables give $e^{-x}I_0(x)$, $e^{-x}I_1(x)$, $e^{-x}\int_0^x I_0(x)dx$, $e^xK_0(x)$, $e^xK_1(x)$, $e^x\int_x^\infty K_0(x)dx$, and e^x to 7S for $x=0(.001)5(.005)15(.01)100$. There are also auxiliary tables to 7S of $I_0(x)$ and $E_0(x)=K_0(x)+\ln xI_0(x)$ for $x=0(.001).15$ and of $I_1(x)$ and $E_1(x)=x\{K_1(x)-\ln xI_1(x)\}$ for $x=0(.001).2$. Interpolation by the three-point Lagrangian method is recommended for small x; linear interpolation is adequate for most of the table. The tables were computed to 10 places on the computer STRELA and it is stated that the errors in the tabulated values do not exceed a unit in the last place. The introduction includes some graphs and summarizes the properties of the functions, giving asymptotic formulas from which values of the functions for $x>100$ can be obtained. The printing and format are satisfactory. *John Todd* (Pasadena, Calif.)

2368:
★**Salzer, H. E. Tables of osculatory interpolation coefficients.** National Bureau of Standards Applied Mathematics Series, No. 56. U.S. Government Printing Office, Washington, D.C., 1959. xi+25 pp. $0.30.

The tables give the coefficients $A_i(p)$ and $B_i(p)$ in the osculatory interpolation formula

$$f(x_0+ph)\sim\sum_{i=-(n-1)/2}^{n/2}\{A_i(p)f_i+B_i(p)f_i'\},$$

for $n=2$, 3, 4 and 5, at interval 0.01 in p. For $n=2$ and 3 the tabular values are exact, for $n=4$ and 5 nine-decimal values are given with an error unlikely to exceed 1.5 units in the last place.

An earlier article [J. Research Nat. Bur. Standards **52** (1954), 211–216; MR **15**, 830, 1140] indicated the advantages of the use of this formula, and this paper gives illustrations of its power for interpolating in tables of Bessel functions and logarithmic functions respectively.

L. Fox (Oxford)

COMPUTING MACHINES

See also 1937, 2559.

2369:
Caracciolo, A.; Cecchini, G.; Gerace, G. B.; Falleni, M.; e Sabbadini, A. Memorie ed entrata-uscita della "macchina definitiva" del C.S.C.E. Nuovo Cimento (10) **12** (1959), 116–122. (English summary)

A description is given of the connections between memories, input-output units and the central part of the computer under construction at the Center of Pisa. The logical design takes into account the possibility of future enlargement of the external units and auxiliary memories of the computer. The possibility has further been considered of direct connections between the Center of Pisa and other scientific centers interested in the use of the computer. *Authors' summary*

2370:
Cecchini, G.; Gerace, G. B.; e Sibani, S. Criteri elettronici di progettazione della calcolatrice del C.S.C.E. I. Nuovo Cimento (10) **12** (1959), 123–125. (English summary)

The criteria adopted in planning the "reduced machine" at present working at the "Centro Studi Calcolatrici Elettroniche" of Pisa University are discussed.

Authors' summary

2371:
Cecchini, G.; e Gerace, G. B. Criteri elettronici di progettazione della calcolatrice del C.S.C.E. II. Nuovo Cimento (10) **12** (1959), 126–129. (English summary)

Experience acquired from the "reduced machine" is described together with some new criteria adopted in planning the "final machine" which is being built at the "Centro Studi Calcolatrici Elettroniche" of Pisa University. *Authors' summary*

2372:
Fabri, E.; e Guerri, L. Impiego della "macchina ridotta" del C.S.C.E. di Pisa nella soluzione di alcuni problemi. Nuovo Cimento (10) **12** (1959), 138–143. (English summary)

A report is given on problems solved or being solved on the "reduced machine" and on some experiments with modern computing techniques. *Authors' summary*

2373:
Eršov, A. P. Programming of arithmetical operators. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 427–430. (Russian)

A description of algorithms for constructing efficient computer commands for automatic execution of arithmetic operations. *L. A. Zadeh* (Berkeley, Calif.)

2374:
Riesenkönig, Wolfgang. Erzeugung lexikographisch geordneter Permutationen in Rechenautomaten. Bl. Deutsch. Ges. Versicherungsmath. **4** (1959), 209–224. (English summary)

"A convenient method of programming digital computers to generate permutations of n elements, in the same order as a dictionary, is introduced and described in detail.

The mathematical basis of the method consists of the introduction of permutation numbers, one number being provided for each permutation.

These permutation numbers provide a theoretically simple method of generating the permutation with any dictionary number selected at will, or conversely determining the dictionary number and sign of a given permutation. Moreover the use of permutation numbers permits the generation of permutations following a given permutation in dictionary order, including their signs, with the least possible number of transpositions. It is shown that, with continuous generation of permutations in dictionary order in accordance with the practical optimum programming method here developed, the average number of transfers required is only 1.24 or 1.31 times, according to the computer employed, what would

be needed for continuous generation by successive transposition. With the latter method, moreover, the principle of dictionary order must be abandoned.

The programming is illustrated by flow diagrams." (Author's summary) *F. Edelman* (Princeton, N.J.)

2375:

★**Tomovic, Rajko. Calculateurs analogiques répétitifs.** Collection de monographies de l'association internationale pour le calcul analogique. Masson et Cie, Paris, 1958. viii+186 pp. 3000 fr.

The field covered in this treatise is essentially that of analog computers adapted to displaying solutions on oscilloscope screens, the calculations being repeated at sweep frequencies. Theoretical discussions summarize the mathematical properties of the problems adaptable to such computers, and of the physical components available for the construction of the computers. Schematics are given for a variety of useful circuits and complete systems are treated briefly.

After an introduction, chapter II on principles discusses the choice of physical quantities for the representation of numbers, the circuit elements that represent particular functions, and the mathematical concepts corresponding to networks of electrical elements.

Chapter III treats the errors of calculation by analog devices, and the stability of such systems in general and of linear systems in particular.

A discussion of scale factors concludes the first section of the book.

A second section on construction starts with an investigation of linear elements and of the regions in which these elements may be considered as linear, with and without feedback.

Under non-linear operations the author describes cathode ray tubes for representing functions, square-law multipliers and commutator and capacitor combinations.

A third section of the book lists and discusses examples of specific differential and integral equations, and the circuits for their representation.

This book views analog computers as existing physical devices whose properties are to be studied, selects a class of these devices for investigation and treats that class in a thorough and useful way.

G. R. Stibitz (Cambridge, Vt.)

2376:

★**Chatterjee, P. N.; and Dutt, D. Uses of an electric potential analogue computer to solve some elasticity problems.** Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 329–340. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

Elastic problems governed by either Laplace's or Poisson's equation for a function $V(x, y)$ are solved by the well known analogue technique of establishing V as the potential of electrically conducting paper. The authors present interesting comparisons with analytical and finite difference methods for the following examples: (I) Torsion of a rectangular prismatic bar; (II) shearing stress distribution in a beam having elliptic cross-section; (III) simply supported equilateral triangular plate subjected to uniformly distributed loading.

H. Bückner (Madison, Wis.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 2358, 2470, 2510, 2526.

2377:

Wunderlich, W. Kinematik in der Ebene der komplexen Zahlen. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 11–18.

Einfache und elegante Anwendungen der komplexen Zahlen in der ebenen Kinematik (Satz von Roberts; die Bahn des Schwerpunktes eines Dreistabgetriebes ist eine Koppelkurve; höhere Radlinien). *O. Bottema* (Delft)

2378:

Horninger, H. Über Trochoidenschraublinien und zyklische Schraubflächen. Monatsh. Math. **63** (1959), 39–58.

Trochoidenschraublinien ergeben sich als Durchdringungskurven koaxialer gerader Kreisschraubflächen und als Bahnkurven von Trochoidschraubungen. Verf. zeigt, dass eine solche Linie auf unendlich viele Arten als Bahnkurve proportional zusammengesetzter euklidischer Schraubungen mit zueinander parallelen Achsen gewonnen werden kann. Jede zyklische Schraubfläche ist durch unendlich viele Trochoidenschraubungen erzeugbar. Betrachtungen über Trochoidenbüschel.

O. Bottema (Delft)

2379:

Šurova, K. E. A fundamental invariant of equations of perturbed motion. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 3, 47–49. (Russian)

The varied (perturbed) motion of a holonomic dynamical system, and the equations which for the motion of such a system follow from Poincaré's equations, are considered. It is shown that Cartan's quadratic form

$$\Omega = \sum_s [\beta_s, \omega_s] - \sum_{(\alpha,\beta)_s} c_{\alpha\beta s}[\omega_\alpha \omega_\beta] y_s$$

is an invariant for the mentioned system of equations, where ω_s denotes the possible displacements (i.e., those which are not violating the constraints), β_s the perturbations (variation of canonical variables $y_s = \partial L/\partial \eta_s$) of the considered dynamical system $c_{\alpha\beta s}$ being structural constants. L is the Lagrangian and the η_α are the actual (concrete) displacements of the system.

This paper is in connexion with a paper of N. G. Četaev [Prikl. Mat. Meh. **19** (1955), 513–515; MR **17**, 795] and can be understood only if one has read a former paper of the author [ibid. **17** (1953), 123–124; **18** (1954), 384; MR **14**, 808; **15**, 997]. *T. P. Anđelić* (Belgrade)

2380:

Grindei, I. Contributions à l'étude de l'intégration des équations du mouvement d'un système mécanique soumis à des liaisons non holonomes. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 81–90. (Romanian. Russian and French summaries)

Pour un système mécanique à n degrés de liberté l'auteur considère des liaisons nonholonomes qui ne sont pas lineaires dans les vitesses et donne les conditions nécessaires et suffisantes pour qu'une réduction à la forme canonique habituelle soit possible. Sur la méthode d'intégration de Hamilton-Jacobi. *O. Bottema* (Delft)

2381:

★Whittaker, E. T. A treatise on the analytical dynamics of particles and rigid bodies: With an introduction to the problem of three bodies. 4th ed. Cambridge University Press, New York, 1959. xiv+456 pp. Paperbound: $4.95.

An almost unchanged reprinting of the fourth edition [Cambridge Univ. Press, 1937].

2382:

Kharlamova, E. I. [Harlamova, E. I.] Rolling of a sphere on an inclined plane. J. Appl. Math. Mech. **22** (1958), 701–708 (504–509 Prikl. Mat. Meh.).

The sphere considered is allowed to be nonhomogeneous with a triaxial ellipsoid of inertia about its center of gravity. But the center of gravity is assumed to be at the geometric center of the sphere. Starting from rest, the sphere rolls without friction on an inclined plane. It is shown that the equations of the problem are similar to those for the rolling of a sphere of this type on a horizontal plane. This reduces the solution of the problem to a previous investigation. *P. Franklin* (Cambridge, Mass.)

2383:

Skowroński, Janisław; and Ziemba, Stefan. The problem of vibrations of non-autonomic systems with strong non-linearity. Arch. Mech. Stos. **10** (1958), 517–523. (Polish and Russian summaries)

Let $x' = F(x, t)$ be a $2s$-dimensional system with F of period L in t and sufficiently smooth to guarantee the existence of a unique solution $x(t)$ through every (x_0, t_0) depending continuously on (x_0, t_0). The authors call such a system of D-class if there exists a finite R such that $\lim\sup_{t\to\infty} \|x(t)\|^2 < R^2$. They prove that every system of D-class has (i) a maximal closed region I invariant under the mapping T: $x(0)\to x(L)$ such that every point of the complement of I (except the point at infinity) tends to I, and (ii) at least one fixed point under T and hence at least one periodic solution of period L.

H. A. Antosiewicz (Los Angeles, Calif.)

2384:

Vorob'ev, L. M. Solution of the basic problem of exterior ballistics. J. Appl. Math. Mech. **22** (1958), 481–492 (350–358 Prikl. Mat. Meh.).

The author adopts a short-arc method for finding two "supporting functions" between whose values those of the desired solution of any given non-linear differential equation must lie. In the two applications here considered, that of a rocket in vacuo, and that of a particle projectile under simple standard conditions, each supporting function over a short interval has its first difference obtained as solution of a nonhomogeneous linear differential equation. The square law of resistance is used to supply auxiliary approximations in the latter problem.

A. A. Bennett (Providence, R.I.)

2385:

Capra, Vincenzo. Sulla traiettoria ottima di un missile leggero soggetto a forza gravitazionale centrale in atmosfera resistente. Boll. Un. Mat. Ital. (3) **13** (1958), 360–371. (English summary)

Consider the motion of a point missile under the assumption that the drag is proportional to a certain power of the velocity, the density of the air varies exponentially with the altitude, and the acceleration of gravity is of the Newtonian type.

The thrust, being a given function of the time, can be oriented at any instant according to an arbitrary program. It is required to prescribe the direction of the thrust so that the horizontal range of the missile is a maximum. It is shown that such a program for the direction of the thrust depends on the solution of a certain functional equation, the numerical solution of which is indicated.

E. Leimanis (Vancouver, B.C.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 2189.

2386:

★Blanc-Lapierre, A.; Casal, P.; et Tortrat, A. Méthodes mathématiques de la mécanique statistique. Masson et Cie, Paris, 1959. x+180 pp. 3800 francs.

This interesting book deals with the same problems as the famous monographs of Khintchin [*Matematičeskie osnovaniya statističeskoi mehaniki*, OGIZ, Moscow-Leningrad, 1943; translated by G. Gamow as *Mathematical foundations of statistical mechanics*, Dover, New York, 1959; *Mathematičeskie osnovaniya kvantovoi statistiki*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951; translated as *Mathematische Grundlagen der Quantenstatistik*, Akademie-Verlag, Berlin, 1956; MR **8**, 187; **10**, 666; **13**, 894], with more emphasis on mathematical completeness and less on physical considerations. In many respects it supplements very nicely Khintchin's monographs. It also can be read independently although one would then miss some of the deep grasp of the subject conveyed by Khintchin's discussion.

The subject matter is standard: properties of phase space (chapter II), ergodic theory (chapter III), mathematical derivation of the canonical distribution for one system from the microcanonical distribution for an ensemble of non-interacting systems. This last question is discussed both for classical and quantum statistics (chapters IV and V respectively). Chapters II and III deal with classical systems only.

Of special interest is the discussion of ergodic theory in chapter III. The authors have analyzed the various mathematical concepts which can be defined in connection with ergodicity, some metric (i.e. referring to the measure in phase space) and some purely topological. The interrelations between these concepts are studied in detail. Proofs are given in a very readable presentation. This chapter can be particularly useful for the physicist who wants a survey of what is known and unknown on the complicated question of ergodicity for classical motion.

The book ends with an appendix giving some additional material on set theory, measure theory and ergodic theorems.

L. Van Hove (Utrecht)

2387:

Smith, E. B.; and Alder, B. J. Perturbation calculations in equilibrium statistical mechanics. I. Hard sphere basis potential. J. Chem. Phys. **30** (1959), 1190–1199.

"A study is made of the effect of potential perturbations on the equilibrium properties of a hard sphere system at both high and low densities. The perturbation is evaluated beyond the first-order term for the second and third virial coefficients to estimate the region of convergence. The results of the first-order perturbation from the virial expansion, superposition theory, and free-volume theory are compared. A comparison with experimental data is made to show the applicability of this perturbation from a hard sphere potential to the estimation of the equilibrium properties of a great variety of gases at moderately high temperatures." (Authors' summary)

H. L. Frisch (Murray Hill, N.J.)

2388:

Morita, Tohru. Cell theory of classical liquids. Phase transition between gas, liquid and solid. III. Yvon's method and its generalization. J. Phys. Soc. Japan **14** (1959), 563–570.

[For parts I and II, see same J. **12** (1957), 1195–1203; MR **19**, 1117.]

This is a cell-cluster theory, based on methods developed by Yvon and Rushbrooke and Scoins. The thermodynamic functions are expressed as multiple lattice sums, and correlations between configurations at different lattice points are approximated by those corresponding to pairs of molecules (as in Bethe's approximation). The theory is worked out for inhomogeneous systems, with the application to phase mixtures in mind. This is developed in the following paper. *H. S. Green* (Adelaide)

2389:

Morita, Tohru. Cell theory of classical liquids. Phase transition between gas, liquid and solid. IV. F.c.c. lattice gases. J. Phys. Soc. Japan **14** (1959), 570–577.

The theory of part III is here applied to a face-centred cubic lattice. The pressure and chemical potential are calculated, in the crude approximation suggested previously, and it is found that the results admit of one phase transition, which is identified with the gas-solid transition. The fact that no liquid state is found is attributed to the fact that too large a lattice distance was chosen, which is perhaps one way of saying that the approximation adopted is inadequate. *H. S. Green* (Adelaide)

2390:

Nagata, Taturo. The cage model by non-spherical interaction potential. Kumamoto J. Sci. Ser. A **3**, 115–123 (1957).

This is an extension of the theory of the Lennard-Jones cell model for liquids to spheroidal molecules. A rectangular lattice with one basic vector shorter than the other is adopted, and formal expressions for the pressure and free energy are derived, but numerical results are not given.

H. S. Green (Adelaide)

2391:

Castoldi, Luigi. Nozione di "albedo" e questioni connesse nella teoria della diffusione dei neutroni termici. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 48–57. (English summary)

Si ristabiliscono e si modificano alcuni risultati classici attorno al significato statistico e alla determinazione sperimentale dell'albedo. *Riassunto dell'autore*

2392:

Costa de Beauregard, Olivier. Équivalence entre les deux principes des actions retardées et de l'entropie croissante. Cahiers de Phys. **12** (1958), 317–326.

This paper proposes to establish a fundamental equivalence between the principle of retarded action or retarded waves on the one hand and the principle of increasing entropy on the other hand. By principle of retarded action or retarded waves the author simply means the rule of solving electromagnetic and quantum-mechanical wave problems with the "outgoing wave condition", i.e., the condition that all perturbed waves produced by interactions are outgoing ones; in other words the rule that the initial value problem must be solved rather than the final value one. The practical validity of this rule results from facts such as the possibility of carrying out a scattering experiment with a plane wave as initial state and the impossibility of carrying out a similar experiment with a plane wave as final state. The same facts, which relate to the practical possibilities in macroscopic experimentation, are responsible for the validity of the principle of increasing entropy. Hence the equivalence stressed by the author. The paper is non-technical. It ends with a remark on quantum field theory, recalling that the use of the so-called causal propagators is a consequence of the outgoing wave condition. *L. Van Hove* (Utrecht)

ELASTICITY, PLASTICITY

See also 2315, 2376, 2433, 2441.

2393:

Seugling, W. R. An iteration method for the solution of finite deformation problems in elasticity. Arch. Mech. Stos. **11** (1959), 3–15. (Polish and Russian summaries)

Using a special stress strain relation author proposes an iteration method for solving finite deformation problems in elasticity. He applies the method to the problem of torsion of cylindrical bars but only considers in detail a circular bar.

(The second solution of the torsion problem for a cylinder of arbitrary section, using a general strain energy function, has been given by R. S. Rivlin [J. Rational Mech. Anal. **2** (1953), 53–81; MR **14**, 513] and W. S. Blackburn and A. E. Green [Proc. Roy. Soc. London. Ser. A **240** (1957), 408–422; MR **19**, 481]. The corresponding problem for an incompressible cylinder was solved by A. E. Green and R. T. Shield [Philos. Trans. Roy. Soc. London. Ser. A **244** (1951), 47–86; MR **13**, 509] and A. E. Green [Proc. Cambridge Philos. Soc. **50** (1954), 488–490; MR **16**, 89]. The third order solution of the torsion problem for a compressible circular cylinder, using a general strain energy function, has been given by A. E. Green and E. W. Wilkes [Quart. J. Mech. Appl. Math. **6** (1953), 240–249; MR **15**, 179].)

A. E. Green (Newcastle-upon-Tyne)

2394:

★Roy, S. K. On the biharmonic analysis of stress fields for hydrostatic loading in relation to analytical relaxational and photoelastic results for openings in structures. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 71–92.

Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

In this paper comparison is made between the analytical, relaxational and photoelastic results for the plane stress problem of a square plate with a symmetrically placed circular hole under hydrostatic pressure. The analytical results are given in detail, using the Airy stress function and polar coordinates, the series solution being limited to a finite number of terms which satisfy the boundary condition for hydrostatic pressure at all points on the inner circular boundary, but only satisfy the required boundary condition on the outer square boundary at a specific number of points. Under these conditions the tangential stresses are found at both boundaries and reference is made to the corresponding results by relaxation and photoelastic methods. The results are compared graphically for several values of the ratio of the side of the square to the diameter of the circular opening.

R. M. Morris (Cardiff)

2395:

Chi, Michael; and Irwin, L. K. Elastic deformations in strips with holes loaded through pins. J. Res. Nat. Bur. Standards **62** (1959), 147–151.

The solution of W. G. Bickley [Philos. Trans. Roy. Soc. London Ser. A **227** (1928), 383–415] for the generalised plane stress in an infinite plate loaded through a pin is combined with the solution of G. Kirsch [Z. Verein. Deutsch. Ing. **42** (1898), 797–807] for a plate with a circular hole at its middle, subject to a uniform unidirectional tension, to give the deformation at any point in an infinite plate loaded through a single pin with an equilibrating force at one end of the plate. For strips whose width is large compared with the diameter of the hole, the effects of the edges of the strip are negligible, and on this basis the above results are applied to a finite strip. In the same way, when two holes are sufficiently far apart the effect on each other is negligible, and the results for the two holes independently are simply superposed. Analytical expressions are derived in this way for strips of width $\geq 14a$ and distance between holes $\geq 8a$, where a is the radius of the hole. Numerical examples are developed and presented in the form of graphs for typical cases. *R. M. Morris* (Cardiff)

2396:

Teodorescu, Petre P. Sur le problème plan de l'élasticité en coordonnées obliques. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 391–410. (Romanian. Russian and French summaries)

En utilisant d'une manière originale certaines composantes des tensions et des déformations spécifiques, l'auteur présente une théorie de l'élasticité plane en coordonnées obliques. On donne les résultats les plus importants pour les tensions et pour les déplacements et les déformations spécifiques. On indique aussi la solution en efforts du problème.

On attaque également quelques problèmes particuliers: le cas des forces massiques quelconques et des forces d'inertie, l'influence de la variation de la température et le cas des poutres-parois anisotropes. On indique en outre quelques méthodes pour résoudre les cas du contour parallélogramme, triangle et trapèze. *Résumé de l'auteur*

2397:

Dutt, S. B. Stress concentrations around a small inclusion (formed by the revolution of a cardioid about its axis) on the axis of a circular cylinder under torsion. Bull. Calcutta Math. Soc. **50** (1958), 29–33.

The title problem has been solved for a rigid inclusion when its axis coincides with that of the cylinder. Modified Bessel's functions have been used and the final results obtained as infinite integrals. *S. C. Das* (Madras)

2398:

Amenzade, Iu. A. Local stresses in the bending of a circular prismatical beam with an elliptical noncoaxial hole. Soviet Physics. Dokl. **122** (3) (1958), 1040–1044 (356–360 Dokl. Akad. Nauk SSSR).

This paper deals with St Venant's flexure problem for the beam described in the title. The stress function is found in the form of an infinite series the coefficients of which are determined from an infinite system of linear equations. The first ten equations of the system are solved by successive approximation and the accuracy of the stress function on the boundaries is checked. Graphical representation of the stress is given for a particular cross section and the position of the centre of flexure found.

R. M. Morris (Cardiff)

2399:

Shirely, L. K.; and Stanišić, M. M. On the torsion of a cylindrical beam having a trapezoidal cross section. J. Aero/Space Sci. **26** (1959), 462–464.

Torsion solutions for a trapezoidal and a triangular section are obtained to a first approximation by the Galerkin method. It is a simple exercise and no references to existing literature are given. *B. R. Seth* (Kharagpur)

2400:

Stănescu, Cristian; et Dumitrescu, Lucian. La torsion des barres cylindriques pourvues d'un canal de coin. Com. Acad. R. P. Romîne **8** (1958), 563–570. (Romanian. Russian and French summaries)

2401:

Manea, V. Torsion de l'arbre à cannelures radiales extérieures. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 423–433. (Romanian. Russian and French summaries)

"On traite, dans ce travail, à l'aide des séries et des systèmes d'équations infinies, le problème de l'arbre à une ou plusieurs cannelures radiales extérieures. On donne l'expression de la rigidité et des tensions et on en fait l'application à deux cas." *Résumé de l'auteur*

2402:

Bassali, W. A. Stresses due to a free circular hole in an infinite thin isotropic plate under certain normal loadings. Bull. Calcutta Math. Soc. **50** (1958), 107–122.

Continuing his previous work on the bending of plates with the help of the complex variable method [Bassali, Proc. Cambridge Philos. Soc. **53** (1957), 728–743, 744–754; **52** (1956), 734–741; MR **19**, 788, 903; Bassali and Dawoud, Mathematika **3** (1956), 144–152; MR **19**, 1212] the author

obtains closed forms of the solutions for perforated infinite plates with free boundaries and subjected to varying normal loads on a circular patch. Two particular cases of concentrated bending and twisting couples are discussed in detail. No numerical results are given.

B. R. Seth (Kharagpur)

2403:

Tamate, Osamu; and Shioya, Shunsuke. On the transverse flexure of a semi-infinite thin plate with a semi-circular notch. Bull. JSME **1** (1958), 357–361.

The stress concentration due to a semi-circular notch in a semi-infinite plate subjected to plain bending is investigated by use of the Poisson-Kirchoff theory of thin plates. The problem is formulated in terms of the complex variable method and a solution for the perturbation due to the notch is sought in the form of a descending power series in $\zeta = re^{i\theta}$. The coefficients of the power series are themselves determined by a perturbation technique based on expanding the coefficients as power series in ε, a function of Poisson's ratio for the plate material.

Numerical calculations are reported for the maximum stress concentration factor due to the notch as a function of Poisson's ratio, and a map of the moment distribution is given for a value of Poisson's ratio of 0.3.

M. Goland (San Antonio, Tex.)

2404:

Gladwell, G. M. L. Some mixed boundary-value problems of aeolotropic thin plate theory. Quart. J. Mech. Appl. Math. **12** (1959), 72–81.

The plates considered have boundaries which are partly clamped and partly free or subjected to given external forces. Details are given for a semi-infinite plate clamped along part of the single straight boundary. This continues an earlier paper [Quart. J. Mech. Appl. Math. **11** (1958), 159–171; MR **20** #3677], in which the author considered the corresponding problems in isotropic plates.

W. R. Dean (London)

2405:

Wilde, Piotr. The general solution for a rectangular orthotropic plate expressed by double trigonometric series. Arch. Mech. Stos. **10** (1958), 747–754. (Polish and Russian summaries)

2406:

Chatterjee, B. B. Note on the bending of a thin uniformly loaded plate bounded by two equal confocal parabolas. Indian J. Theoret. Phys. **6** (1958), 35–37.

2407:

Soare, Mircea. Die Membrantheorie der Schalen mit zwei Leitparabeln und einer Leitebene. Rev. Méc. Appl. **2** (1957), no. 1, 213–232.

Es wird die Membrantheorie einer Schale abgeleitet, deren Mittelfläche die Form einer Fläche mit zwei Leitparabeln und einer Leitebene besitzt. Es werden die Schnittkräfte für Eigengewicht, Schneelast, Flüssigkeitsdruck und einigen allgemeineren Belastungsfällen untersucht. Es werden in bekannter Weise die Gleichgewichtsbedingungen durch eine Spannungsfunktion erfüllt. Die Spannungsfunktion läßt sich in diesem Fall aus einer partiellen linearen und nicht homogenen Differentialgleichung erster Ordnung bestimmen. Am Schluß der Arbeit wird ein Zahlenbeispiel für ein rechteckig begrenztes Dach unter Eigengewicht vorgeführt.

W. Zerna (Hannover)

2408:

★Aas-Jakobsen, A. Die Berechnung der Zylinderschalen. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xii + 160 pp.

Das Buch behandelt die Berechnung der Zylinderschalen in umfassender Weise. Im ersten Abschnitt werden die grundlegenden Gleichungen für isotrope Kreiszylinderschalen und für Schalen mit zentrisch und exzentrisch angeordneten Ringrippen behandelt. Anschließend wird eine strenge Lösung für das kreiszylindrische Rohr gegeben, wobei die äußere Belastung, die Schnittkräfte und Verschiebungen in Fourierreihen entwickelt werden. Die Membrantheorie des Rohres wird aus dem ersten Reihenglied gewonnen. Für verschiedene Belastungen werden die Membrankräfte angegeben.

Die Untersuchung der an den Längsrändern belasteten offenen Kreiszylinderschale wird in der üblichen Weise durchgeführt, die auf die bekannte charakteristische Gleichung achter Ordnung führt. Für Isotrope und für Schalen mit Ringrippen werden Tabellen angegeben, die es ermöglichen eine genaue Berechnung der auftretenden acht Integrationskonstanten durchzuführen.

Zur schnellen Berechnung der Tonnendächer schlägt der Verfasser ein Verfahren vor, welches er die allgemeine Balkenmethode nennt. Zu diesem Zweck führt er eine sogenannte Modellschale ein und entwickelt ein Verfahren, das so aufgebaut ist, daß keine Gleichungen mit mehreren Unbekannten zu lösen sind. Die errechneten Schnittkräfte der Modellschale werden auf die wirklichen Schalen übertragen, wobei der Verlauf in Ringrichtung erhalten bleibt, während für den Verlauf in der Längsrichtung für die Längs- und Schubkräfte die Verteilung der elementaren Balkentheorie angesetzt wird. Auf diese Weise kann auch eine Durchlaufwirkung in der Längsrichtung berücksichtigt werden.

Weiter enthält das Buch ein Näherungsverfahren für nichtkreiszylindrische Schalen, einen Abschnitt über die Bewehrung der Schalen und die Berechnung der Binder, eine Erörterung der Stabilitätsfragen, sowie eine kurze Behandlung der vorgespannten Tonnen.

Zahlreiche Rechenbeispiele veranschaulichen in allen Abschnitten die theoretischen Ausführungen und erläutern den Gebrauch der Tabellen. Für die praktische Berechnung von Zylinderschalen stellt das Buch eine wertvolle Hilfe dar.

W. Zerna (Hannover)

2409:

Chattarji, P. P. On the stresses in twisted composite spherically isotropic spheres. Bull. Calcutta Math. Soc. **50** (1958), 99–102.

Stresses in closed forms have been obtained in the case of two different spherically isotropic materials twisted by couples.

S. C. Das (Madras)

2410:

★Crandall, Stephen H. (Editor). Notes for the M.I.T. special summer program on random vibration. The

Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass., 1958. viii+406 pp.

This book represents notes prepared for a special summer program on random vibrations held at M.I.T. from June 23 to July 3, 1958. The first six chapters treat basic concepts and background material. Chapter 1 contains a review of ordinary vibration theory presented in a form which facilitates the extension to random excitation. The fundamental statistical notions involved in random processes are introduced in chapter 2 and a wide range of examples of mechanical random processes described in chapter 3. In chapter 4 the response of a vibratory system to random excitation is examined. In chapter 5 the known facts regarding structural damping are surveyed and in chapter 6 current theory and practice regarding fatigue failure summarized. The last six chapters apply the material developed in the first six to the problems of design and testing. *W. Freiberger* (Providence, R.I.)

2411:

★**Christiansen, Jerald N. Transient motion in mechanical wave-filters.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 117–121. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

The author has given a most interesting theoretical derivation of a general class of mechanical wave filters and demonstrates his theory by application to physical problems of which one is the beat phenomenon. This occurs when the coupling between oscillators is slight. The first section contains the general derivation of n oscillators for the steady state condition. The second and third sections consider the effect of transients. The transients are very aptly handled by application of the Fourier Transfer approach, which is one of the prevalent modern day approaches to transient solutions. The reviewer believes that an alternative approach would be the mechanical impedance method [*Colloquium on mechanical impedance methods for mechanical vibrations*, American Society of Mechanical Engineers, New York, N.Y., December, 1958], and for purposes of calculation, digital computers would be simpler to apply. The author has given a most interesting solution to the general problem of mechanical wave filters.

H. Saunders (Philadelphia, Pa.)

2412:

Safronov, Yu. V. Stability of orthotropic rectangular plates with two rigidly fixed edges. Akad. Nauk Ukrain. RSR. Prikl. Meh. 4 (1958), 285–293. (Ukrainian. Russian and English summaries)

2413:

Grigolyuk, È. I. The stability of nonhomogeneous elastoplastic shells. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 663–666. (Russian)

2414:

★**Miklowitz, Julius. On the use of approximate theories of an elastic rod in problems of longitudinal impact.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 215–224. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A number of approximate theories of wave propagation in elastic cylinders are considered and their predictions in the problem of longitudinal impact are compared. It is shown that for large times they all lead to similar results and are in fair agreement both with the predictions of the exact theory and with the results of experiment. Of the approximate theories considered, that due to Mindlin and Hermann is found to give the closest agreement at large times. *H. Kolsky* (London)

2415:

★**Jahsman, W. E. Propagation of abrupt circular wave fronts in elastic sheets and plates.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 195–202. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

The author considers a ring load applied suddenly to an elastic, homogeneous, isotropic sheet or plate. The velocities and amplitudes of the wave fronts which spread inwards and outwards from the ring as concentric circles are studied by the method of characteristics.

Equations of motion which allow for shear and rotary inertia are used. It is shown that in terms of these equations there are two velocities of the wave fronts in a sheet, c_1 and c_2 (the irrotational and equivoluminal velocities, respectively, for an unbounded elastic solid), whereas in a plate there are three such distinct velocities, c_2, κc_2 and c_p, where κ is the shear correction factor used by Mindlin and $c_p=\sqrt{(E/\rho(1-\nu^2))}$ (E is Young's modulus, ν Poisson's ratio and ρ the density). The amplitudes of all abrupt circular wavefronts vary inversely as the square root of the radius. *H. Kolsky* (London)

2416:

Karal, Frank C., Jr.; and Keller, Joseph B. Elastic wave propagation in homogeneous and inhomogeneous media. J. Acoust. Soc. Amer. **31** (1959), 694–705.

The authors apply a method developed in electromagnetic theory which gives an asymptotic series solution of initial and boundary value problems. The linearized equations of motions of elasticity are solved formally and approximately by means of asymptotic series whose first term gives the "geometrical optics" effect. The media can be homogeneous or inhomogeneous. To be specific, we mention the homogeneous case. The vector particle displacement vector u satisfies the linearized equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}.$$

Solutions are sought of the form $\mathbf{u}=\mathbf{A}\exp\{i\omega(S(x,y,z)-t)\}$ where $\mathbf{A}=\sum_{n=0}^{\infty}(i\omega)^{-n}\mathbf{A}_n$. The method determines S and the coefficients $\mathbf{A}_n$ of the series. Two types of waves arise in elastic wave propagation, compressional and shear. For each of these there are a function S and coefficients $\mathbf{A}_n$. The functions S are solutions of the first order partial differential equation known in optics as the eiconal equation and the functions $\mathbf{A}_n$ are solutions of first order linear, ordinary differential equations which express the variation

of A_n along the rays; these are the orthogonal trajectories of the family of surfaces $S=\text{const}$.

The theory for inhomogeneous media is the same in principle but more complicated analytically because the basic partial differential contains more terms. Also the ordinary vector differential equation for $\mathbf{A}_n$, in the case of both shear and compressional waves, is a coupled system in the components of $\mathbf{A}_n$. But a solution as a vector equation is possible. The initial conditions for S and the $\mathbf{A}_n$ must be determined by the source.

M. Kline (New York, N.Y.)

2417:

Skuridin, G. A. The Duhamel principle and asymptotic solutions of the dynamical equations of the theory of elasticity. I. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 3–10. (Russian)

The author determines an asymptotic expansion in inverse powers of the frequency for the displacement vector in an elastic medium which satisfies the dynamical equations of linear elasticity theory where a sinusoidal time dependent body force is prescribed. The method follows closely the work of M. Kline [Comm. Pure Appl. Math. **4** (1951), 225–262; MR **13**, 408], who treats the analogous problem in electromagnetic theory.

R. A. Toupin (Washington, D.C.)

2418:

Green, A. E.; Rivlin, R. S.; and Spencer, A. J. M. The mechanics of non-linear materials with memory. II. Arch. Rational Mech. Anal. **3** (1959), 82–90.

As an extension of the work on constitutive equations for materials with memory initiated in earlier papers [Green and Rivlin, same Arch. **1** (1957/58), 1–21, 470; MR **20** #2130], the authors examine materials in which the stress has an arbitrary polynomial dependence on these quantities at previous times τ_α ($\alpha=1, 2, \cdots, N$) in the interval ($0<\tau_\alpha<t$). Attention is focussed on bodies which are initially isotropic and alternative forms of the stress deformation relations for these are derived. As in the earlier work, the procedure leads to an expression for the stress in terms of functionals each of which is approximated by the sum of a number of multiple integrals. The resulting forms are reduced by making use of results derived by Spencer and Rivlin [ibid. **2** (1958/59), 309–336; MR **20** #7030] in the theory of symmetric matrix polynomials.

J. E. Adkins (Nottingham)

2419:

★**Lessen, Martin. On the motion of a thermo-viscoelastic solid.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 20–28. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

The assumption of entropy functions, which with strain tensors determine in power series form the internal energy in the elastic elements of the standard linear viscoelastic solid, permits stress-strain-temperature relations to be developed. These are specialised for Maxwell and Voigt (Kelvin) behavior, and equations for the propagation of longitudinal waves are deduced.

The statement that mass is associated with elastic elements only does not seem significant to the reviewer, since at a point only one continuum velocity arises. Nor does the relative microscopic nature of part of the viscoelastic model seem necessary, for the corresponding strain tensors will not satisfy compatibility conditions since they are not representable in terms of displacements in the usual way.

E. H. Lee (Providence, R.I.)

2420:

Paria, Gunadhar. Deformation of a porous viscoelastic body containing a fluid under steady pressures. Bull. Calcutta Math. Soc. **50** (1958), 71–76.

A viscoelastic porous body containing a fluid, viscoelastic in dilatation, is subject to external forces applied at time zero. The four viscoelastic stress-strain laws required are all of the form appropriate to the three-element elastic model. All dependent variables are functions of the cylindrical radial coordinate, r, and of the time. When the normal solid and fluid pressures are given on the boundary $r=a$, the first term of the series expansion for the displacement at $r=a$ is found.

D. R. Bland (Manchester)

2421:

Budiansky, Bernard. A reassessment of deformation theories of plasticity. J. Appl. Mech. **26** (1959), 259–264.

It is generally accepted today that constitutive laws for isothermal plastic deformation of work-hardening solids are intrinsically of the "rate type", establishing a homogeneous relation between the rates of stress and plastic strain. This relation depends on the state of the solid, which may for instance be specified by the stress history leading to the instantaneous state of stress. Drucker's quasi-thermodynamic definition of work-hardening imposes certain conditions on the mathematical structure of these "rate laws" [Proc. 1st U.S. Nat. Congr. Appl. Mech., Chicago, 1951, pp. 487–491, Amer. Soc. Mech. Engrs., New York, 1952; MR **14**, 929].

For a given stress history, a rate law associates a definite plastic strain with the final state of stress, but another stress history terminating in the same state of stress may lead to another final plastic strain. It has nevertheless been suggested that constitutive laws of the "finite type" establishing a definite relation between stress and plastic strain may yield useful results in many problems of plastic stress analysis. The present paper treats the following problem, which is of obvious interest in this connection: given a "finite law" and a specific stress history, to determine whether there exists a rate law satisfying Drucker's conditions that will predict the same development of plastic deformation for this particular stress history as the given finite law. In treating this problem, the author exploits the following remarks: (1) the given finite law does not restrict the choice of the yield surface associated with the instantaneous state of the solid; (2) for the study of continued plastic flow only the local geometry of the yield surface at the instantaneous stress point needs to be known; (3) at a regular point of the yield surface (i.e. point with unique tangent plane), Drucker's conditions are more restrictive than at a singular point. For a particular finite law (Nadai's law) the conditions are established under which an equivalent rate law exists for a given stress history. {It should be noted that the investigation is essentially concerned with homogeneous states of stress and plastic strain. For a homogeneous solid a stress analysis based on a finite law would be justified

only if the same equivalent rate law applied to the stress histories of all particles. On the common yield surface for the virgin state of all particles, singular points must be the exception. Though a stress history starting from a regular point may lead to a neighboring yield surface with a singularity at the instantaneous stress point, this singularity cannot be pronounced. Accordingly, there are severe restrictions on the stress histories in the neighborhood of the virgin yield surface, and the practical usefulness of the author's criteria is not clear.}

W. Prager (Providence, R.I.)

2422:

Paul, Burton. Collapse loads of rings and flanges under uniform twisting moment and radial force. J. Appl. Mech. **26** (1959), 265–270.

A circular ring is acted upon by a uniform twisting moment M and a uniform radial force Q around its outer circumference. Treating the ring as a curved beam, the author finds the interaction curve against plastic collapse for a ring of arbitrary cross section. The particular case of a ring of uniform section is also treated by plate theory; the results are in good agreement if the inner and outer radii are almost equal. The results are used to find the plastic collapse load of a corrugated tube under an axial load.

P. G. Hodge, Jr. (Chicago, Ill.)

2423a:

Storchi, Edoardo. Linearizzazione del problema plastico ristretto degli sforzi piani. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 173–205.

2423b:

Storchi, Edoardo. Sulla risoluzione del problema plastico ristretto degli sforzi piani. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 321–348.

For plane plastic problems the author defines ω and ϕ by $\omega=(\sigma_{xx}+\sigma_{yy})/4K$ and $\phi \sin 2\theta=(\sigma_{xx}-\sigma_{yy})/4K$, where θ is the angle between either direction of maximum shearing stress and the y-axis. The yield criterion is expressed in the form $\phi=\phi(\omega)$. In the first paper the two equilibrium equations are transformed into differential equations for the coordinates x and y in terms of ω and θ. In plane stress, both for Tresca's criterion and for a new criterion introduced by the author, these equations can be integrated. In the second paper further transformations of dependent and independent variables reduce both equations to the form

$$\frac{\partial^2 f(\alpha,\beta)}{\partial\alpha\partial\beta}+g(\alpha,\beta)f(\alpha,\beta)=0,$$

where $g(\alpha,\beta)$ is dependent upon the yield criterion. This reviewer is of the opinion that these transformations will remain mathematical curiosities until they are used to solve specific practical problems in plasticity.

D. R. Bland (Manchester)

2424:

Ivlev, D. D. Approximate computation by the small parameter method for plane elasto-plastic problems in the theory of ideal plasticity. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 5, 17–26. (Russian)

Analysis is given for the situation described in the title. Essentially, the procedure is to represent stresses and strains as power series in a small parameter δ. A highly-complex system of sets of equations is then derived: each set corresponding to one particular power of δ. Application is made to the progressive, approximate solution of certain illustrative problems in which δ is defined in terms of the geometry of a natural boundary.

H. G. Hopkins (Sevenoaks)

2425:

Schumann, Walter. On isoperimetric inequalities in plasticity. Quart. Appl. Math. **16** (1958), 309–314.

The main result of this paper is that $P \geqq 6\pi M_0$ where P is the total limit load and M_0 the yield moment of a thin, perfectly plastic, simply supported, uniformly loaded plate of arbitrary shape. In other words, the circular plate has a smaller limit load than any other plate of equal area. The argument consists in the main of mapping the arbitrary region into a circle and showing that the rate of energy dissipation is not thereby increased. In conjunction with the upper bound theorem of limit analysis, the result follows.

W. E. Boyce (Troy, N.Y.)

2426:

Romiti, Ario. Soluzioni discontinue nei problemi di equilibrio limite dei materiali dotati di coesione ed attrito interno. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 395–410.

Discontinuous fields of stress in two-dimensional problems of soil mechanics are discussed with the aid of a "shock polar". In particular, the character of the active and passive stress fields in the neighborhood of a point of discontinuity of the surface traction is investigated. While some of the results were already given by J. Massau [*Mémoire sur l'intégration graphique des équations aux dérivées partielles*, G. Delporte, Mons, 1952; MR **14**, 588; p. 267 ff.], the present complete discussion covers several new cases.

W. Prager (Providence, R.I.)

2427:

Dvořák, Jan. Kreisringplatte in elastisch-plastischem Zustande. Apl. Mat. **4** (1959), 18–34. (Czech. Russian and German summaries)

Die Lösung des Spannungszustandes an einem Rande frei gestützter, am anderen durch gleichmässig verteilte Kraft belasteter Kreisringplatte. Es wird ein isotroper Werkstoff, der in plastischem Zustande keine Verfestigung aufweist, vorausgesetzt. Die Plastizitätsbedingung wird mit der Hypothese von Huber-Mises-Hencky gegeben.

Die Gleichungen des elastisch-plastischen Gleichgewichts werden mit dem Näherungsverfahren gelöst. An einem numerischen Beispiel wird gezeigt, dass bereits die erste Näherung zufriedenstellend genaue Ergebnisse gibt. Auf diese Weise besteht die Möglichkeit die Verläufe gesuchter Grössen (Biegungsmomente, Durchbiegung, Grenze zwischen der elastischen und der plastischen Zone) in geschlossener Form zu bestimmen.

Zusammenfassung des Autors

2428:

Zadoyan, M. A. A variational equation in the nonlinear theory of creep. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 277–282. (Russian. Armenian summary)

The principle of complementary energy and Clapeyron's theorem are derived for a non-linear material described by a stress-strain relation in the form of a hereditary integral. The results obtained represent a particular case of the known variational theorems of the mechanics of inelastic continua [A. M. Freudenthal and H. Geiringer, *Handbuch der Physik*, vol. 6, pp. 229–433, Springer, Berlin-Göttingen-Heidelberg, 1958; MR **20** #483; see p. 256].

A. M. Freudenthal (New York, N.Y.)

2429:

Gaponov, V. V. On displacements of soils which are in a state of limiting equilibrium. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 65–74. (Ukrainian. Russian and English summaries)

The displacements associated with limiting equilibrium solutions are found by numerical integration using Il'yushin's stress-strain relations for total strains [A. A. Il'yushin, *Plastičnost'. Čast' pervaya. Uprugo-plastičeskie deformacii*, OGIZ, Moscow-Leningrad, 1948; MR **12**, 373]. The dilatation is assumed to be wholly elastic.

Numerical values are given for a battered retaining wall with surcharge.

R. M. Haythornthwaite (Ann Arbor, Mich.)

2430:

Forray, M. J. Thermal stresses in plates. J. Aero/Space Sci. **25** (1958), 716–717.

The stress distribution in a plate subjected to an arbitrary temperature field is found as the solution of the inhomogeneous biharmonic equation $\nabla^4\phi = -E\alpha\nabla^2 T$ subject to $\phi = \partial\phi/\partial n = 0$ on the boundary. The solution is developed in series using Fourier expansion of temperature T in terms of the polar angle. An example for a circular bulkhead with non-radially symmetric heating is given.

E. H. Mansfield (Farnborough)

2431:

Forray, Marvin. Thermal stresses in rings. J. Aero/Space Sci. **26** (1959), 310–311.

Author applies general theory [see review above] to determine thermal stresses in an annulus with arbitrary temperature distribution.

E. H. Mansfield (Farnborough)

2432:

Shevchenko, Yu. N. Temperature stresses in a thick-walled cylinder with a change in the modulus of elasticity along its generatrix. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **4** (1958), 401–410. (Ukrainian. Russian and English summaries)

An isotropic thick-walled cylinder of finite length in an axisymmetric steady temperature field is considered, assuming that the variation of αT along the radius ρ is represented by polynomials and in the longitudinal direction ζ by exponential functions (α = coefficient of thermal expansion, T = temperature). The shear modulus μ varies only along the ζ-direction according to an exponential law.

The solution is split into two parts: (1) A general solution for isothermal case with variable μ; in this case the displacement and stress are represented in a form similar to that of Grodsky, by means of a stress function which approximately satisfies the equation $\nabla_1^2\nabla_1 f = 0$, where

$$\nabla_1^2 = \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2} + b\frac{\partial}{\partial\zeta},$$

(b = given constant); (2) a particular solution for the thermoelastic case.

A numerical example is solved, the boundary conditions on the bases of the cylinder being satisfied in an integral manner.

J. Nowinski (Madison, Wis.)

STRUCTURE OF MATTER

See also 2148c.

2433:

Maradudin, Alexei A. Screw dislocations and discrete elastic theory. Phys. and Chem. Solids **9** (1958), 1–20.

The continuum approximation to the atomic displacements of a screw dislocation leads to logarithmically divergent energies associated with the dislocation core. By assuming discrete atoms having Hooke's-law interactions with nearest neighbours only it is possible to make a mathematically tractable model not suffering from this difficulty. The detailed calculation is carried through for two atomic arrangements, corresponding to two possible dislocation axes associated with slip on {110} planes of alkali-halide crystals, and leads to strain energies of 5.36 and 8.09 electron-volts per atomic plane for the two cases.

A. J. C. Wilson (Cardiff)

2434:

Schmidt, Paul W.; and Hight, Robert, Jr. Calculation of the intensity of small-angle x-ray scattering at relatively large scattering angles. J. Appl. Phys. **30** (1959), 866–871.

FLUID MECHANICS, ACOUSTICS

See also 2348, 2475, 2574.

2435:

Manea, V. Calcul de la masse apparente pour une carène allongée. Rev. Méc. Appl. **2** (1957), no. 2, 89–96.

The virtual mass for bodies of revolution generated by sources and sinks is calculated, and approximations are given for elongated bodies in terms of the virtual mass for bodies in plane flow. It is observed that the latter can be expressed in terms of a conformal mapping onto the unit circle.

P. R. Garabedian (New York, N.Y.)

2436:

Sabaneev, V. S. The induced masses of an oval ellipsoid moving in a fluid bounded by a plane wall. Vestnik Leningrad. Univ. **13** (1958), no. 19, 170–186. (Russian. English summary)

"The paper considers the unsteady motion of an ellipsoid in an ideal incompressible fluid bounded by a plane wall. The induced masses are calculated using the approximate expressions for the velocity potentials given by Eisenberg [P. Eisenberg, J. Appl. Mech. **17** (1950), 154–158; MR **12**, 58] and the author.

The induced mass coefficients are calculated for all cases where the semimajor axis of the ellipsoid keeps parallel to the wall all through the motion. The values of these coefficients for the different ellipsoids are presented in diagrams.

The obtained approximate solutions are compared with the exact ones in two limiting cases, in the case of sphere and in that of circular infinite cylinder. All the approximate solutions approach the exact ones in the limit as the ellipsoid recedes from the wall." *Author's summary*

2437a:

Morton, B. R. Forced plumes. J. Fluid Mech. **5** (1959), 151–163.

2437b:

Morton, B. R.; Taylor, Geoffrey; and Turner, J. S. Turbulent gravitational convection from maintained and instantaneous sources. Proc. Roy. Soc. London Ser. A **234** (1956), 1–23.

The second article is an extension of the earlier investigation in the first article, of plumes from a source of buoyancy only to plumes involving release of mass, momentum and buoyancy. The plume flow is supposed to be turbulent and therefore Gaussian profiles of mean vertical velocity and buoyancy are assumed. An inflow speed having a constant ratio to the vertical speed at plume axis is also assumed. The fluid is incompressible. First, forced plumes from a point source of buoyancy and momentum are treated; it is pointed out that the mass flow from such a source must vanish. For forced plumes in a uniform environment the results are compared with a pure momentum-source (jet) and buoyancy-source (plume), and several different possible behaviors are described. It is next shown that a forced plume in a uniform environment, from a finite area, involving mass release, can be related to a virtual point source.

Proceeding to forced plumes in a stably stratified environment, the authors replace the Gaussian profiles by "top hat" profiles, on the basis of the earlier investigation [loc. cit.]. Since the stratification affects appreciably only the upper part of the plume, the relationship between finite-area and point-source plumes can be carried over, and only point-source plumes need be studied here. Again various behaviors are described for various combinations of parameters. The paper includes a concise review of previous theoretical and experimental work, and data on the various empirical constants of the theory are referred to. *W. R. Sears* (Ithaca, N.Y.)

2438:

★Stoker, J. J. Water waves: The mathematical theory with applications. Pure and Applied Mathematics, Vol. IV. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1957. xxviii+567 pp. $12.00.

The need for a modern book on the theory of water waves has long been recognized and it is not an easy task to supply one, in part due to the large variety of applications to specific problems in hydrodynamics (in the wide sense of the word) and in part due to the fact that from the mathematical point of view, the study of water waves draws freely from other disciplines—notably the general notions of diffraction theory in acoustics and electromagnetic theory.

The book can best be regarded in two fashions. First we may consider the content of the book as one in classical theoretical physics. In a leisurely manner, we are shown the fundamental equations and the approximations to these, upon which most of the physical problems in the book are based. Then we may consider the numerous problems to which the approximate theories apply. For example, waves on sloping beaches and past obstacles, waves on a running stream and ship waves, long waves in shallow water (the breaking of a dam, tides, floating breakwaters, etc.), mathematical hydraulics (flood prediction). This is but a sample of the numerous problems which are discussed and we should be grateful to the author for permitting, as he tells us, his "tastes and predilections free rein" in this aspect of his writing. This cross section of the book will be a source of information on the subject of water waves for many years to come.

The mathematical methods employed run through many phases of linear methods in classical analysis, that is, linear partial differential equations (Laplace's and the Helmholtz equation), function theory, Fourier and Laplace transforms, etc., as well as some non-linear methods. It appears that all the tools have been laid out in neat order and need only be used. The development of the subject has proved otherwise and in fact many of these problems require subtle use of these mathematical concepts in order to gain effective solutions to the problems at hand.

In writing a book of this type, no criticism should fall on the author if he omits some of the specific physical problems or some of the useful mathematical methods. The book was not intended to be a treatise on water waves. Indeed the author tells us that he has written a book which is rather personal in character and the material is very often chosen simply because it interests him. Furthermore, he tells us that the book has a personal flavor from another point of view since much of the work is based on the work of the staff of the Institute of Mathematical Sciences of New York University. It is the feeling of the reviewer that in writing the book from this perfectly legitimate point of view, the author has nevertheless dismissed the related work of others in an overly summary fashion. This would be particularly true, for example, of his discussion of the method of Wiener and Hopf. The reviewer does not agree with the author's opinion, unsupported by any evidence, that the method of Wiener and Hopf can in general be replaced by simpler methods. Indeed, one has the feeling that the author does not take kindly to this method and is perhaps not aware of its full potentiality. Incidentally, the integral equation (5.6.7) is incorrect [see C. J. Bouwkamp, Rep. Progr. Phys. **17** (1954), 35–100; MR **16**, 200].

The 12 chapters of the work are divided into four parts as follows: (1) basic hydrodynamics; the two basic approximate theories; (2) waves simply harmonic in the time; motions starting from rest; transients; waves on a running stream; shock waves; (3) long waves in shallow water; mathematical hydraulics; (4) problems in which the surface conditions are satisfied exactly; the breaking of a dam; Levi-Civita's theory.

A. E. Heins (Ann Arbor, Mich.)

2439:

Jolas, Pierre; et Kravtchenko, Julien. Remarques sur la théorie des ondes liquides de gravité de Nekrassov. C. R. Acad. Sci. Paris **248** (1959), 2550–2552.

Faisant suite à la Note ci-dessous, les Auteurs explicitent, jusqu'au 3ème ordre, les équations paramétriques de la surface libre et la célérité. Ils montrent l'équivalence complètent des résultats de Nekrassov et de Levi-Civita, la différence ne portant que sur le choix du petit paramètre. Les processus de ces deux auteurs sont comparés, quant à la simplicité des raisonnements de convergence, et à la rapidité des approximations successives. En conclusion, la variante de la méthode de Nekrassov, présentée par les Auteurs, combinée avec la notion de majorante stokienne de Levi-Civita, qu'ils utilisent, est celle qui conduit le plus rapidement au résultat. Notons d'ailleurs que plusieurs difficultés, qu'aucune des méthodes ci-dessus ne parait susceptible de résoudre, n'ont été surmountées que récemment par R. Gouyon [Thèse, Toulouse, 1958]. *R. Gerber* (Grenoble)

2440:

Jolas, Pierre; et Kravtchenko, Julien. Remarques sur la théorie des ondes liquides de gravité de Nekrassov. C. R. Acad. Sci. Paris **248** (1959), 2440–2442.

Les auteurs présentent une variante des raisonnements de Nekrassov [*Točnaya teoriya voln ustanovivšegocya vida na poverhnosti tyaželoi židkosti*, Izdat. Akad. Nauk SSSR, Moscow, 1951; MR **15**, 659] qui a l'avantage de simplifier le processus d'approximation, et d'en montrer l'équivalence complète avec les méthodes de Levi-Civita. Dans cette note, les Auteurs indiquent leur méthode de calcul, dans le cas de la profondeur infinie, et en faisant, comme Nekrassov, l'hypothèse de la symétrie des ondes. Leur méthode, comme celle de ce dernier auteur, repose sur l'emploi d'un petit paramètre, mais la résolution de l'équation intégrale, non linéaire, de Nekrassov est remplacée par celle d'un système intégro-différentiel. Les calculs s'en trouvent beaucoup simplifiés. Les valeurs numériques des coefficients du développement du potentiel des vitesses sont données, jusqu'au troisième ordre, avec le paramètre choisi par Nekrassov. Quelques erreurs numériques du calcul de cet Auteur ont été rectifiées.

R. Gerber (Grenoble)

2441:

Dumitrescu, Lucian; et Stănescu, Cristian. Un problème aux limites et ses applications au mouvement de deux fluides visqueux en contact, et à la torsion d'une barre non homogène. Rev. Méc. Appl. **2** (1957), no. 2, 79–88.

The authors consider the following two problems: (1) the uniform viscous flow of two fluids in a cylindrical pipe against a constant pressure gradient, (2) the torsion of a cylindrical bar consisting of two materials. In both cases, the common boundary of the two fluids (materials) is a plane which is parallel to the generators of the cylinder. Neglecting the inertia term in the first problem, the authors show that both cases depend upon the solution of Laplace's equation in the cross-sectional plane. Specific solutions are given for proper boundary data when the cross section is either circular or rectangular. The method of solution involves the use of Green's function, the condition that an unknown harmonic function and its normal derivative be equal on the common boundary to another unknown harmonic function and its normal derivative, and the solution of an integral equation.

N. Coburn (Ann Arbor, Mich.)

2442:

Azpeitia, Alfonso Gill; and Newell, Gordon Frank. Theory of oscillation type viscometers. IV. A thick disk. Z. Angew. Math. Phys. **10** (1959), 15–34. (German summary)

[For earlier parts, see J. Kestin and Newell, same Z. **8** (1957), 433–449; MR **19**, 1218; D. Beckwith and Newell, ibid. 450–465; MR **19**, 1218; Azpeitia and Newell, ibid. **9a** (1958), 97–118; MR **20** #4989.]

This paper is the fourth in a series dealing with the theory of oscillation type viscometers. It deals with a method for computing the edge effects for a disk performing torsional oscillations in the case that the boundary layer thickness is small compared with both the thickness of the disk and the radius of the disk. The case of the thickness of the disk much less than the boundary layer thickness which in turn is much less than the radius of the disk is treated in paper III.

The method of analysis is similar to that used in III. A Wiener-Hopf technique which involves some rather interesting use of complex variable theory is used to obtain a first approximation for the edge effects on the viscous drag. This result is then improved by using a variational procedure that was developed in III. The results are sufficiently accurate to allow the use of the oscillating disk for precision absolute measurements of viscosity.

Finally the results of this paper coupled with the results of III are used to provide interpolation formulas for the drag of a disk of arbitrary shape provided the boundary layer thickness is small compared to the radius.

R. C. DiPrima (Troy, N.Y.)

2443:

Newell, Gordon Frank. Theory of oscillation type viscometers. V. Disk oscillating between fixed plates. Z. Angew. Math. Phys. **10** (1959), 160–174. (German summary)

This fifth paper deals with an oscillating disk placed between fixed plates in such a manner that the separation between the plates is small compared to the radius of the disk and also the boundary layer thickness. If the latter were reversed, we would have the cases considered in III and IV [see above review]. Using these assumptions it is shown that the partial differential equation (in dimensionless variables) governing the velocity distribution can be approximated quite accurately by Laplace's equation including the situation at the edge of the disk. This problem is then solved by conformal mapping techniques. An approximation to the viscous drag exerted by the fluid on the disk is computed and improved by the use of a variational principle [see above review]. The results agree quite well with experimental data.

R. C. DiPrima (Troy, N.Y.)

2444:

Hocking, L. M. A doublet technique for solving Oseen's equations of axisymmetric flow. Mathematika **5** (1958), 134–140.

Solutions for the Oseen equations are constructed by

superposition of elementary 'doublet' solutions for which the stream function ψ is given by

$$\psi = (2k)^{-1}(1+\cos\theta)[1-\exp\{kr(\cos\theta-1)\}]$$

where $k=u/2\nu$. It is shown that for a given flow outside a region S, the doublet distribution in S is not determined uniquely, but that uniqueness can be achieved by imposing an auxiliary condition, the simplest being that the total strength of the distribution is zero. The method is illustrated by constructing the known solution for Oseen flow past a sphere. *O. M. Phillips* (Baltimore, Md.)

2445:

Yang, Kwang-Tzu. Unsteady laminar boundary layers over an arbitrary cylinder with heat transfer in an incompressible flow. J. Appl. Mech. **26** (1959), 171–178.

The problem of the title is approached by the integral procedure. Based on the tabulated values by the author [same J. **25** (1958), 421–427] for problems with an undisturbed velocity of the form $x/(1-at)$, a set of universal functions is calculated. The reliability and limitation of these functions are discussed in the light of several simple problems, solutions of which are available.

L. N. Tao (Chicago, Ill.)

2446:

Meksyn, David. Sur la position des singularités dans les solutions de l'équation de la couche limite. C. R. Acad. Sci. Paris **248** (1959), 2286–2287.

L'A. si occupa delle singolarità delle soluzioni della equazione

$$f''+ff'' = \lambda(1-f'^2),$$

nel campo complesso. Supponendo che le soluzioni siano funzioni meromorfe (ipotesi questa che il recensore non sa quanto sia giustificata) l'A. dimostra che esse hanno solo poli del 1° ordine e dà uno sviluppo che mette in evidenza le singolarità, tenendo conto di simmetrie che presentano le soluzioni stesse. L'A. accenna in modo vago alla possibilità di calcolo approssimato dei poli. Gli errori di stampa sono numerosi. *G. Prodi* (Trieste)

2447:

von Baranoff, Alexis. Sur une solution exacte de l'equation d'Orr-Sommerfeld. C. R. Acad. Sci. Paris **248** (1959), 186–188.

It is shown how the usual form of the Orr-Sommerfeld equation for two-dimensional plane flows can be transformed into an equivalent Volterra integral equation of the second kind. A formal solution of this integral equation is obtained in the form of a power series in the Reynolds number. This form of solution is not a practical one for the many problems in which the critical Reynolds number is large and, for such cases, the author suggests the use of an iterative procedure. *W. H. Reid* (Providence, R.I.)

2448:

Rabenstein, Albert L. The determination of the inverse matrix for a basic reference equation for the theory of hydrodynamic stability. Arch. Rational Mech. Anal. **2** (1958/59), 355–366.

This paper is a continuation of the author's earlier work [same Arch. **1** (1958), 418–435; MR **20** #3337] on the equation

$$u^{iv}+\lambda^2(zu''+\alpha u'+\beta u) = 0, \tag{1}$$

which serves as a reference equation for the Orr-Sommerfeld equation. From the solutions of (1) it is possible to construct in a formal way an asymptotically exact solution of the Orr-Sommerfeld equation. In the proof of this statement it becomes necessary to know the asymptotic behavior of the inverse matrices corresponding to the fundamental matrix solutions of (1). For a given fundamental matrix, the elements of the inverse matrix have different forms for different sectors of the complex z-plane, and formulae are given for the third order Wronskians of all fundamental solutions from which the corresponding inverses may be found.

W. H. Reid (Providence, R.I.)

2449:

Witting, Hermann. Über den Einfluss der Stromlinienkrümmung auf die Stabiltät laminarer Strömungen. Arch. Rational Mech. Anal. **2** (1958), 243–283.

Assuming a two-dimensional laminar flow with wavy streamlines, the author investigates the local stability of the flow with respect to small disturbances in the form of vortices with their axes in the main-stream direction (Taylor-Görtler vortices). Two such flows are examined, each of which is a slightly distorted Blasius boundary layer, the distortion being due to a wavy wall in the first case and to Tollmien-Schlichting waves in the second. The main conclusion is that in the valleys of the waves the flow is unstable if the streamline curvature is large enough. The corresponding critical amplitude of the Tollmien-Schlichting waves in the second example is of the order of 10^{-4} boundary-layer thickness. The possible relevance of the results to the problem of laminar-turbulent transition is discussed. The paper gives considerable attention to several of the standard assumptions and approximations used in problems of this type. In particular, for the evaluation of higher-order curvature effects, a perturbation procedure is developed in which the eigenfunction and eigenvalue are expanded in terms of a parameter proportional to the streamline curvature. The eigenvalue problem is solved at each stage of the perturbation expansions by means of the integral-equation approach previously used by Görtler [NACA Tech. Memo. no. 1375 (1954); MR **15**, 999] and Hämmerlin [J. Rational Mech. Anal. **4** (1955), 279–321; MR **16**, 876].

D. W. Dunn (Ottawa, Ont.)

2450:

Kampé de Fériet, J. La notion de moyenne dans la théorie de la turbulence. Rend. Sem. Mat. Fis. Milano **27** (1957), 167–207.

"Definitions and conditions for average values are indicated in order to transform Navier equations of viscous flow into Reynolds turbulence equations. A discussion shows that classical average definitions are inadequate. The right definition is investigated with the help of topological ring theory. Finally statistical mean definition is discussed." (Author's summary)

Shu-Teh chen Moy (Detroit, Mich.)

2451:

Dulov, V. G. Decay of an arbitrary initial discontinuity

of the gas parameters across a jump-discontinuity in the cross-sectional area. Vestnik Leningrad. Univ. **13** (1958), no. 19, 76–90. (Russian. English summary)

Uniform states of flow are prescribed on either side of the area step at $t=0$, viz., u_0, p_0, ρ_0 and u_1, p_1, ρ_1; one-dimensional perfect-gas dynamics then permits to introduce shocks and/or simple waves to describe the decay of the initial jump of the state parameters. The decay is assumed to establish finally a (nearly) steady flow in the neighborhood of the step, so that the usual three conservation laws can be applied between two sections slightly upstream (2) and downstream (3). Therefore, the pressure on the step surface p^1 must be assumed, namely as $p_2+\alpha(F_2/F_3,\ M_2)\rho_2 u_2^2$ for final outflow (into the smaller area (3)), or as a function of the values of u, p, ρ at (2) and (3) for final inflow, to make the problem determinate. In general, the steady flow regime begins to grow from $t=0$ on, and the initial discontinuity is swept along with the steady flow. The complete discussion leads to 10 and 8 possible transient configurations for final outflow and inflow respectively and to the determination of regions in the p_1/p_0, u_1/u_0-plane which permit the realization of each of these configurations.

{Reviewer notes that the term "decay of discontinuity" is misleading, since in a few of the configurations the initial pressure jump either remains constant or even increases.} *G. Kuerti* (Cleveland, Ohio)

2452:

Ray, M. Note on the variation of temperature due to small steady disturbances in a compressible flow. Bull. Calcutta Math. Soc. **50** (1958), 150–154.

Earlier incorrect solutions [see same Bull. **45** (1953), 45–49; MR **15**, 662] are replaced by certain approximate ones. *D. C. Pack* (Glasgow)

2453:

Čulkin, P. I. Computation of the flow around an arbitrary profile and a solid of revolution in subsonic flow of a gas (symmetric case). Vyčisl. Mat. **3** (1958), 99–110. (Russian)

Let ξ, η be ellipsoidal coordinates with foci within the prescribed profile $t=t_1(\eta)$, where $t=e^{-\xi}$. The equations of continuity and irrotationality become

$$-\partial(t\chi)/\partial t+\partial\omega/\partial\eta+2\chi=0$$
$$\partial(t\lambda)/\partial t+\partial\mu/\partial\eta-2\lambda=0$$

where $\chi=h\rho u$, $\omega=h\rho v$, $\lambda=hv$, $\mu=hu$ and u, v are ξ, η velocity components, and $2h=(1-2t^2\cos 2\eta+t^4)^{0.5}$. Now integrate both partial differential equations with respect to t from $t_0=0$ to $t_n(\eta)=N^{-1}(N-1-n)t_1(\eta)$ $(n=1, 2, \cdots, N)$. Approximate each of χ, ω, λ, μ by a polynomial of the form $F(t,\ \eta)=\sum_1^{N+1} a_m(t/t_1)^m$. Then

$$\int_0^{t_n} F\,dt = t_1(\eta)\sum_0^N b_{nt}F(t_t(\eta),\ \eta),$$

where b_{nt} are known constants. In this way the $2N$ integrals yield a system of ordinary differential equations approximately satisfied by $u_t=u(t_t(\eta),\ \eta)$ and v_t. The system is completed by the boundary condition on the body and one boundary condition at infinity; the second boundary condition at infinity is used for an accuracy check. Certain constant coefficients arising in the application of this method have been tabulated for $N=1, 2, 3$. A similar method is described for axisymmetric flow. Numerical results are given and compared with results obtained from other sources for flows about Joukowski airfoils and ellipsoids of revolution. *J. H. Giese* (Aberdeen, Md.)

2454:

Kackova, O. N. Computation of annular supersonic jets and diffusors. Vyčisl. Mat. **3** (1958), 111–129. (Russian)

Suppose that in the annular region $x\leqq 0$, $a\leqq r\leqq 1$ there is uniform flow at sonic speed. Attempt to extend the central body (with discontinuous tangent at 0, a) a finite distance in such a way as to produce eventually (without shocks) uniform flow in the tube $0\leqq r\leqq 1$. Numerically this can be done by a straightforward method of characteristics calculation, provided one can find approximate values of flow functions and coordinates on one of the characteristics from the corner A to the outer wall, close to $x=0$. To find these initial data the author transforms the equations for irrotational flow into a form with flow inclination θ, Mach angle α, x, and r as dependent variables; and stream function z and a characteristic variable χ as independent variables. Since $x(z,\ \chi)$ can be found after θ, α, and r are known, the author expands

$$\alpha=\tfrac{1}{2}\pi+\sum_1^a \alpha_n(z)\chi^n,$$

$$\theta=\sum_1^a \theta_n(z)\chi^n,\quad r=z+\sum_1^a r_n(z)\chi^n,$$

and describes a method to calculate coefficients up to α_2, θ_4, and r_5 inclusive, though only α_1, θ_3, and r_3 are non-zero. Coordinates and θ and α distributions have been tabulated for calculations of seven central bodies, corresponding to exit Mach numbers from 1.9 to 3.2.

J. H. Giese (Aberdeen, Md.)

2455:

Carafoli, E.; et Ionescu, M. Écoulements coniques d'ordre supérieur autour des ailes triangulaires minces ou à épaisseur symétrique. Rev. Méc. Appl. **2** (1957), no. 1, 5–27.

Cet article est consacré à l'exposé de la théorie générale des écoulements coniques d'ordre supérieur, qui sont aussi souvent appelés écoulements homogènes. Après avoir obtenu les relations de compatibilité pour les fonctions analytiques dont les parties réelles définissent les dérivées d'ordre n du potentiel des vitesses, les auteurs procèdent à l'analyse des singularités. Ils étudient les solutions correspondantes et développent alors la théorie générale des ailes applaties sur un angle pour des répartitions polynomiales de la déflection. Pour résoudre un problème spécifié, il faut résoudre un système d'équations linéaires; aucune méthode générale n'est proposé pour obtenir une résolution systématique de ce problème.

P. Germain (Paris)

2456:

Carafoli, Elie; et Horovitz, Béatrice. Écoulements coniques d'ordre supérieur autour d'une aile triangulaire à plaque normale. Rev. Méc. Appl. **2** (1957), no. 2, 5–19.

La théorie des écoulements homogènes est appliquée à

l'étude d'une aile en delta munie d'une plaque normale triangulaire symétrique à bords subsoniques. L'aile et la plaque peuvent comporter des arêtes partant du sommet commun. La composante verticale de la vitesse de perturbation sur l'aile et la vitesse latérale sur la plaque sont des polynomes homogènes donnés. Dans ce mémoire sont calculées les contributions à l'écoulement des différents bords. On en déduit le calcul de la pression sur l'aile et la plaque. Une application complète est faite dans le cas d'un mouvement homogène d'ordre 2.

P. Germain (Paris)

2457:

Carafoli, Elie; et Năstase, Adriana. Étude des ailes triangulaires minces à symétrie forcée dans un courant supersonique. Rev. Méc. Appl. **3** (1958), no. 4, 373–391.

Cette étude, comme la précédente [#2455 ci-dessus] est effectuée dans le cadre des applications de la théorie des mouvements homogènes. Les problèmes étudiés sont essentiellement relatifs à des ailes portantes à bords d'attaque subsonique, admettant le plan Ox_1x_3 comme plan de symétrie, et situées au voisinage du plan Ox_1x_2. Par hypothèse, la composante normale de la vitesse de perturbation sur l'aile est soit un polynome en x_1, x_2, pair en x_2, soit un polynome en x_1 et x_2 ne contenant que des puissances impaires de x_2 dans la région $x_2>0$ et le polynome opposé dans la région $x_2<0$. C'est ce dernier cas que les auteurs appellent "symétrie forcée". Le principe du calcul de la pression est exposé de façon générale et les cas des mouvements homogènes d'ordre 2, 3, 4 sont considérés en détail. Des compléments sont fournis sur le calcul des efforts globaux et celui des forces de succion. Le problème traité ici a déjà été envisagé par G. N. Lance [Aero. Quart. **6** (1955), 149–163; MR **16**, 971] et M. Fenain et D. Vallée [C. R. Acad. Sci. Paris **244** (1957), 1138–1141; MR **19**, 1224]; dans ce dernier travail les formules de résolution sont données pour toute valeur de l'ordre d'homogénéité.

P. Germain (Paris)

2458:

Zel'dovich, Ia. B. Cylindrical self-similar acoustical waves. Soviet Physics JETP **6** (1958), 537–541.

Zababakhin and Nechaev [#2467 below] investigated the propagation of a weak cylindrical shock acoustic wave and its reflexion from the axis. They found that the reflected wave solution yields a pressure which diverges logarithmically on the wave front. This situation does not occur for spherical convergent waves. In this paper the author has studied the whole family of the so-called self-similar solutions of a cylindrical shock wave. The results of Zababakhin and Nechaev are a special but important case of solutions which are derived here. The procedure is briefly as follows. Starting with a plane wave in one dimension, the author constructs the solution of the wave equation for the pressure in cylindrical coordinates in terms of an integral expression based on the superposition principle of plane waves. By expressing the kernel in the form of a power (n) of the argument $z=r\cos\varphi-ct$, the pressures, in front of and behind the shock, of the incident wave focused on the axis and of the reflected wave are expressed in terms of an integral over the angular parameter $k=\cos\varphi$. Near the front the pressure increases gradually when the exponent $n>-\frac{1}{2}$, but for $n=-\frac{1}{2}$ a finite discontinuity occurs at the front. For $n<-\frac{1}{2}$, the pressure becomes infinite. The case $n=-\frac{1}{2}$, which was treated by Zababakhin and Nechaev, shows that the pressure behind the wave front drops with a finite slope away from the front; it is independent of the position of the front. To obtain the unique solution for the reflected pressure, the author makes use of the superposition principle (plane waves) which admits no singularities in crossing the axis at the initial time $t=0$. This leads to the results already derived by Zababakhin and Nechaev provided the pressure thus derived is not equal to the absolute gas pressure in which the wave propagates. The acoustic approximation used in deriving the pressure after reflexion is valid so long as the distance from the focal point is greater than a constant quantity which depends inversely on the square of the gas pressure. In this case the equality of the pressure profile in front of and behind the shock front is not preserved.

N. Chako (Flushing, N.Y.)

2459:

Chernov, L. A. The acoustics of a moving medium. Survey. Soviet Physics. Acoust. **4** (4) (1958), 311–318 (299–306 Akust. Ž.).

This is a short survey of the problems of sound propagation in moving media, as well as the problems involving moving sound sources and receivers. The emphasis is on listing and giving very short descriptions of important papers in the field. There is no attempt to present a comprehensive view of the field or to outline a program for the future.

I. Stakgold (Washington, D.C.)

2460:

Serrin, James. The derivation of stress-deformation relations for a Stokesian fluid. J. Math. Mech. **8** (1959), 459–469.

The author gives a neat, partly expository derivation of the now familiar general form of constitutive equations for a fluid such that the stress depends only on velocity gradients and scalar thermodynamic variables. Neglecting the latter, this form is $T=\alpha I+\beta D+\gamma D^2$ where T, I and D are respectively the stress, unit and rate of deformation matrices, while α, β, γ are scalar functions of the principal invariants of D. The main new result is that α, β, γ are continuous functions of their arguments when $T(D)$ is thrice continuously differentiable, but not necessarily when $T(D)$ is merely continuous. The author glosses over some subtleties revealed by the example $T=dD$, where d is the maximum principal value of D. For example, in principal coordinates, each component of T is then a polynomial in the components of D, so the hypothesis of Theorem 2 might be construed as holding. The conclusion fails. Theorem 2 is true if the hypothesis holds in every coordinate system.

J. L. Ericksen (Baltimore, Md.)

2461:

Rikitake, Tsuneji. Magneto-hydrodynamic oscillations of a perfectly conducting fluid sphere placed in a uniform magnetic field. J. Phys. Soc. Japan **13** (1958), 1224–1230.

An attempt is made to improve theories of magneto-hydrodynamic oscillations of an incompressible fluid sphere with a rigid boundary in a uniform magnetic field. Unlike the previous theories which have been developed by M. Schwarzschild [Ann. Astrophys. **12** (1949), 148–160]

and the author [Bull. Earthquake Res. Inst. Tokyo **33** (1955), 175–198] analytical expressions for zonal oscillations are obtained rigorously in this paper. It turns out that no eigen-periods are determined accurately. However, they are approximately obtained with drastic simplifications. Two fundamental normal modes of oscillations are also approximately obtained and illustrated. (From the author's summary) *K. C. Westfold* (Sydney)

2462:

Shankaranarayana Rao, B. A note on rotating configuration associated with toroidal magnetic field. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 315–318.

"The equilibrium form of an incompressible fluid mass rotating about an axis in presence of a toroidal magnetic field, whose axis coincides with the axis of rotation, is discussed and the conditions that the equilibrium configuration may be a sphere, a prolate spheroid or an oblate spheroid are derived." (Author's abstract)

H. Greenspan (Cambridge, Mass.)

2463:

Crupi, Giovanni. Sulla velocità di gruppo nella magneto-idrodinamica. Boll. Un. Mat. Ital. (3) **13** (1958), 539–542.

This paper considers the relationship between group velocity and phase velocity of magneto-hydrodynamic waves, propagating in an incompressible fluid, which is electrically conductive and moves in an external magnetic field of flux density B_0. Starting from a previously deduced [same Boll. **12** (1957), 604–609] formula for the phase velocity W_f, the calculation of group velocity W_g is straightforward and leads to the result:

$$W_g = \frac{W_f}{1-a},$$

where a is a positive number inferior to unity. Some special cases are considered. *M. J. O. Strutt* (Zürich)

2464:

Staniukovich, K. P. Some stationary relativistic flows. Soviet Physics. Dokl. **119** (3) (1958), 299–303 (251–254 Dokl. Akad. Nauk SSSR).

The flows under consideration are adiabatic stationary flows with symmetry. The basic equations of these flows are

$$u\frac{d(wu)}{dr}+\frac{dw}{dr}=0, \quad u\frac{d\lg v}{dr}=\frac{du}{dr}+\frac{Nu}{r},$$

$$\sigma=\text{const}, \quad w=w(v).$$

(u = 4-velocity; $w=\rho v+\rho vc^2$ = heat content; w_0 = heat content at rest; v = specific volume; σ = entropy, $N=0, 1, 2$, respectively, for one-dimensional, cylindrical and spherical flows). The solution of these equations is

$$\frac{w}{\Theta}=\frac{w}{\sqrt{(1-a^2/c^2)}}=w_0,$$

$$\frac{ar^N}{v\Theta}=\frac{ar^N}{v\sqrt{(1-a^2/c^2)}}=\text{const}=m=\frac{\bar{m}}{\bar{B}}$$

($\bar{B}=1, 2\pi, 4\pi$, respectively for $N=0, 1, 2$; a = velocity of the medium). Knowing the equation of state the functions $a(r)$ and $w(r)$ can be determined. The author discusses the properties of the flow for $r=r_{\min}$ and $r\to\infty$ ($pv^k=$ const.). For an extreme relativistic gas $v\to\infty$, $w\to 0$, $a\to c$ for $r\to\infty$. For an ordinary not extreme relativistic gas $w\to c^2$, $v\to\infty$, $(a=c(1-c^4/w_0^2))$ for $r\to\infty$. The author gives a numerical example for spherical motion of an extreme relativistic gas ($N=2$). Finally the author discusses the statistics and the thermodynamic relations of an extreme relativistic (photon) gas.

M. Pinl (Cologne)

2465:

Frankl', F. I. Potential steady relativistic gas flows. Dokl. Akad. Nauk SSSR **123** (1958), 47–48. (Russian)

Eindeutig bestimmte relativistische Gasströmungen wurden bereits von K. P. Stanyukowich betrachtet [cf. K. P. Stanyukowich, das vorstehende Referat]. In der vorliegenden Note handelt es sich um den allgemeinen Fall einer eindeutigen räumlichen relativistischen Potentialströmung und speziell um besondere einfache Fälle ultrarelativistischer Gase.

Im Falle einer reibungslosen, eventuell jedoch wärmeleitenden, barotropen Strömung existiert ein Strömungspotential. Die Größen

$$v^i=\frac{w}{\rho}u^i=Yu^i$$

bezeichnet Verfasser als Komponenten der "Pseudogeschwindigkeit". Dabei ist u^i der Vektor der relativistischen Vierergeschwindigkeit, ρ und w bedeuten normierte Masse-Energie-bzw. Wärme-funktionsfaktoren mit $Y=w/\rho\geq 1$.

Im Falle einer Potentialströmung gilt $v_i=\partial\varphi/\partial x^i$ und das Potential der Pseudogeschwindigkeit genügt der Differentialgleichung

$$[g^{ik}+(\bar{a}^{-2}-1)u^iu^k]\frac{\partial^2\varphi}{\partial x^i\partial x^k}=0.$$

Dabei bedeutet $\bar{a}$ die relativistische Schallgeschwindigkeit. Im Falle eindeutig bestimmter Gasströmungen gilt $\partial v_i/\partial x_0=0$ oder $\partial v_0/\partial x^i=0$ ($i=0, 1, 2, 3$) oder auch $v_0=$ const. als relativistische Bernoullische Gleichung. Die gewöhnliche Geschwindigkeit, d.h. der Dreiervektor $\hat{u}^i=dx^i/dx^0$ wird dann durch die Gleichungen $\hat{u}_i=-v_i/v_0$ ($i=1, 2, 3$) festgelegt. Auf diese Weise ergibt sich für die gewöhnliche Geschwindigkeit $\hat{u}_i$ der betrachteten Strömungen ein Potential, das mit $\hat{\varphi}$ bezeichnet wird. Für

$$g_{00}=-g_{11}=-g_{22}=-g_{33}=1, \quad g_{ik}=0, \quad i\neq k$$

wird die Differentialgleichung angegeben, der $\hat{\varphi}$ genügt. Im Falle ultrarelativistischer Gase gilt $\bar{a}^2=\frac{1}{3}$ und die Potentialgleichung fällt mit der klassischen Potentialgleichung für ideale Gase zusammen, wenn man den Wärmeleitungskoeffizienten $k=2$ setzt. Zum Schluß erwähnt Verfasser noch den Fall eines Photonen-gases, in welchem $\rho=0$ gilt. *M. Pinl* (Cologne)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 2149, 2503, 2506, 2507.

2466:

Sumi, Masao. Theory of spatially growing plasma waves. J. Phys. Soc. Japan **14** (1959), 653–657.

This paper is a continuation of the same author's work on plasma waves [same J. **13** (1958), 1476–1485]. It deals

with the fundamental equations as a boundary particle problem. These equations are Fourier analysed in time and Laplace transformed in space, and a dispersion relation is written down for the growing waves in the forward direction. The equations are solved numerically and graphs are given for the growth rate and phase velocity as a function of the thermal velocity of the plasma electrons, of the beam velocity, and of the plasma density. Experiments by Boyd, Field, and Gould [Phys. Rev. (2) **109** (1958), 1393–1394] are in qualitative agreement with the predictions. *M. J. Moravcsik* (Livermore, Calif.)

2467:

Zababakhin, E. I.; and Nechaev, M. N. Electromagnetic-field shock waves and their cumulation. Soviet Physics JETP **6** (1958), 345–351.

In this article the authors have investigated the properties of the electromagnetic shock wave produced by a shock wave emitted by a perfect conducting substance which fills the entire half-space and is separated from the rest (vacuum space) by the plane $x=0$. The vacuum space is under the influence of a steady magnetic field H_0, normal to the plane. It is assumed that the plane surface separating the two media suddenly attains a velocity u in the direction of the field. By using coordinates attached to the moving plane boundary, the solutions of the fields E and H, derived from Maxwell equations, are expressed in terms of the stationary magnetic field (H_0), the factor δ representing the increase in density of the conducting substance due to the shock wave, the conductivity λ of the substance, and the parameter $q=x-Dt$ (D is the velocity of steady wave propagation), both in vacuum and in the substance, as well as the current density in the vacuum space. The field quantities E and H and the current density j contain a parameter $L=(c^2-D^2)/(4\pi\lambda D)$ (c = velocity of the electromagnetic wave). The perturbation of the magnetic field is found to lead the shock wave by an effective length l, which, on the boundary separating the media, gives rise to an electromagnetic wave in the vacuum with the same time spread or width of the front, that is, equal to 14.8 cms, for copper at room temperature. In copper $l=2.5\mu$. Here, D is equal to 5×10^5 cms/sec. At low temperature the width is considerably smaller, due to the higher conductivity of the copper. On the other hand, the total current associated with the shock wave is independent of the conductivity, and it vanishes behind the shock front.

For the case of convergent cylindrical waves, the amplitudes of the field vary inversely as the square root of the radius of the wave front. These expressions are valid for both small and large amplitudes of the cylindrical wave. To obtain self-similar solutions for convergent cylindrical waves, the authors have expressed the fields in terms of functions with argument $\tau=ct/r$, where t is measured at the instant of focusing. The results of calculations show that the fields are finite for $\tau<1$ and diverge for $\tau\geq1$; i.e., the amplitude of the reflected wave is unbounded not only on the axis, but also at some distance from the axis, a situation which does not occur for acoustic waves, except on the axis itself. It should be pointed out that these results hold for both compression and rarified waves. For $1<\tau<\infty$, i.e., solutions behind the wave reflected from the axis, convergent solutions are obtained for points not too close behind the wave front. However, the sum of the two solutions representing the fields E and H, respectively, is free of divergences on the wave front. On the axis E approaches zero and the magnetic field H decreases as $t^{-1/2}$ and approaches the finite value H_0 on the axis. The analytical results are illustrated by a number of graphs.

N. Chako (Flushing, N.Y.)

2468:

Klose, Wolfgang. Wärmeleitfähigkeit dünner Drähte. Z. Naturf. **13a** (1958), 978–985.

The thermal conductance of a metallic conductor decreases when the geometrical dimensions of the conductor become of the order of magnitude of the mean free path of the conduction electrons calculated from the Bloch theory. Physically the effect is interpreted as an increase in the number of interactions between conduction electrons and the walls of the conductor. After the onset of the effect the conductance can be made to increase again by the application of a magnetic field which decreases the number of interactions. The present paper gives a theory of the effect based on the Boltzmann equation and an approximate expression for the interaction between electrons and wall. The interaction term is discussed in detail and a parameter is introduced which corresponds to a relaxation time. *B. Gross* (Rio de Janeiro)

2469:

Lebedev, N. N. The electric field at the edge of a plane condenser containing a dielectric. Soviet Physics. Tech. Phys. **28** (3) (1958), 1234–1243 (1330–1339 Z. Tehn. Fiz.).

Problems in electrostatics involving different dielectric media are of known complexity. The present one, by the introduction of a new boundary condition, is solved by two infinite Fourier integral representations. Some boundary conditions are excepted from this solution. The solution leads to a pair of coupled integral equations. The kernel of one of these equations is expressed by means of gamma functions and is studied asymptotically. The solution of the coupled integral equations is obtained as the limit of a function, defined in the complex plane by means of this kernel. Insertion into the original Fourier integrals yields the solution of the electrostatic problem as two infinite definite integral expressions. The residues of the integrands are studied, and from these the final solution is then obtained. Some special cases are considered, and numerical results are given (table).

M. J. O. Strutt (Zürich)

2470:

Harrison, E. R. Epicyclic orbits of charged particles. Amer. J. Phys. **27** (1959), 314–317.

The author obtains the possible orbits of a charged particle which moves in a uniform magnetic field $\mathbf{B}$, in a plane perpendicular to this field, and subject also to a central force of the form $\mathbf{F}=\xi\mathbf{r}$, where ξ is a positive or negative constant. The mathematical problem is to obtain solutions of the differential equation

$$\ddot{\mathbf{r}} = \xi m^{-1}\mathbf{r} - 2\dot{\mathbf{r}}\times\boldsymbol{\omega},$$

where m is the mass of the particle, e is its charge, and $\boldsymbol{\omega}=-e\mathbf{B}/2m$; the solutions sought are further restricted to those lying in the equatorial plane of the central force.

The possible orbits are all epicyclics provided $\xi < m\omega^2$, where ω is the magnitude of $\boldsymbol{\omega}$. The types of these epicyclics are described. *M. Kline* (New York, N.Y.)

2471:

Wilcox, Calvin H. Spherical means and radiation conditions. Arch. Rational Mech. Anal. **3** (1959), 133–148.

The author seeks to supply a more natural foundation for the Sommerfeld radiation condition, which has served as a boundary condition to insure the uniqueness of solutions of the Helmholtz or time-free wave equation, wherein harmonic time behavior of the solution is understood. He wishes then to characterize uniquely solutions of the problem $\Delta u(p)+k^2u(p)=f(p, k)$, k constant, and $u(p)=0$ or $\partial u(p)/\partial n=0$ on the boundary S of the exterior domain B in which the differential equation is to hold. (An exterior domain is one which contains all points outside of some sphere.)

By using some theorems in spherical means of solutions of the homogeneous Helmholtz equation $\Delta u(p)+k^2u(p)=0$ he shows that $u(p)=u_1(p)+v_1(p)$ for p in B, where

$$u_1(p) = (4\pi)^{-1}\int_{S'}\{(\partial u/\partial n)(e^{ikR}/R)-u\partial(e^{ikR}/R)/\partial n\}dS$$

and S' is any surface containing p and S and where

$$v_1(p) = (4\pi)^{-1}\int_{S}\{(\partial u/\partial n)(e^{ikR}/R)-u\partial(e^{ikR}/R)/\partial n\}dS.$$

The author then shows that the condition $u_1\equiv 0$ is a natural radiation condition. He does this by considering an initial-boundary value problem which corresponds to the boundary value problem being solved, namely,

$$\Delta\phi(p, t)-(\partial^2/\partial t^2)\phi(p, t) = f(p, k)e^{ikt},$$

for p in B and $t>0$ with the initial conditions $\phi(p, 0+)=0$, $\partial\phi(p, 0+)/\partial t=0$ for p in B and one of the boundary conditions $\phi(p, t)=0$ or $\partial\phi(p, t)/\partial n=0$ on S for $t>0$. By introducing what has been called the pulse problem, which is essentially the same initial-boundary value problem but with e^{ikt} replaced by $\delta(t)$, one can show that $\phi(p, t)=u(p, t)+e^{-ikt}u(p)$. Here $u(p, t)$ is a transient; that is, the limit as $t\to\infty$ of $u(p, t)=0$. The natural radiation condition is used to show that $u(p)$ is the unique solution of the boundary value problem described above. This condition must of course be added to the boundary conditions.

The natural radiation condition is equal to

$$\int_{S(p,r)}\{\partial(ru)/\partial r-ikru\}dS = 0$$

for all p and for all r larger than the maximum distance from p to S. It is then shown to be equivalent to several other radiation conditions in use including the Sommerfeld condition. The equivalence of several radiation conditions for vector diffraction problems, that is, boundary value problems involving Maxwell's equations, is also demonstrated. *M. Kline* (New York, N.Y.)

2472:

★Демков, Ю. Н. Вариационные принципы в теории столкновений. [Demkov, Yu. N. Variational principles in the theory of collisions.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 168 pp. 5.85 rubles.

This book represents some basic formulations of variational principles in the theory of collisions. The variational approach is a new one and finds slowly its application both in the theory and numerical calculations. The topics treated by the author are: general formulation of variational principles, relationships between various formulations of variational principles and their application to the theory of collisions, symmetry of functionals, principle of a detailed balance and unitarity of scattering operator, scale variations and virial theorem for the scattering problem.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

2473:

Gutmann, Marcian. Un nouveau critère élémentaire de stabilité et son extension à une classe de fonctions de transfert, transcendentes uniformes. Bul. Inst. Politehn. Bucureşti **19** (1957), no. 3/4, 37–43. (Romanian. Russian and French summaries)

If $F(p)$ is the driving point admittance or impedance a linear system, where p is the complex frequency of $p=s+jw$, the following stability criterion is established: $d/dw\,[\arg F(p)]<0$ $(p=jw)$. *B. Gross* (Rio de Janeiro)

2474:

★Wagner, T. C. Gordon. Analytical transients. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xi+202 pp. $8.75.

This book discusses methods for the determination of transients of linear systems. The treatment is primarily mathematical and in some cases includes discussion of problems of convergence. Nevertheless, sufficient emphasis is laid on an adequate presentation of the physical basis of each problem treated and on the physical significance of the choice of method. This applies particularly to the systematic discussion of initial conditions and the way in which the choice of conditions is connected with the mathematical treatment. Fourier, Laplace, and complex variable methods for the determination of transients and complex variable methods for the determination of stability conditions are given. Applications are taken mostly from electrical circuit theory, but a short discussion of electro-mechanical analogies and d'Alembert's and Lagrange's principles shows the formal equivalence of electrical and mechanical systems. The book deals mostly with problems described by total differential equations; partial differential equations are discussed in the final chapter in connection with complex variable theory. The book covers a wide field and is recommended as a comprehensive text on modern methodology in the field. Contents: Mathematical introduction; Methods of network analysis; Initial transient behavior; Fourier analysis and related topics; Stability theorems; Additional properties and applications of the Laplace transformation; Applications of complex variable theory and the inversion integral. *B. Gross* (Rio de Janeiro)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

2475:

Hassan, H. A. On heat transfer to laminar boundary layers. J. Aero/Space Sci. **26** (1959), 464.

2476:

Usol'cev, S. A. Solution of the heat-conduction equation for a semi-infinite rod with discontinuous coefficient. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. 6 (10) (1957), 82–86. (Russian. Kazah summary)

Consider the first boundary problem for the heat equation $u_t = a(x)u_{xx}$ on $0<x<\infty$, $0<t\leq T$, where $a(x)=a_1^2$ for $0<x<x_1$, $=a_2^2$ for $x_1<x<x_2$, $\cdots$, $=a_{n+1}^2$ for $x_n<x$. This is equivalent to the first boundary problem for $v_t=v_{xx}$, with piece-wise continuous data on $t=0$ and with certain compatibility conditions for v and v_x on the lines $x=x_i$ $(i=1, \cdots, n)$. The author shows by means of Fourier sine transforms that the latter problem can be reduced to a system of Volterra integral equations.

D. G. Aronson (Minneapolis, Minn.)

2477:

Duffin, R. J. A variational problem relating to cooling fins. J. Math. Mech. 8 (1959), 47–56.

The author considers an interesting problem in the calculus of variations in which one varies not only over solution functions but also over coefficient functions. Existence and uniqueness of the solution are demonstrated, and the precise analytic form is derived.

R. Bellman (Santa Monica, Calif.)

2478:

Černigovskaya, E. I. Heat waves in a layer and in a slab lying on the layer. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 9, 91–93. (Russian)

The author considers the problem of finding the temperature distribution in a layer in which the upper boundary is subjected to a periodic source. A second problem, for which the first appears as an auxiliary, is concerned with the finding of the temperature in a slab lying on the layer. The boundary function dictates the type of solution which is assumed so that the problem is reduced to the solution of a set of ordinary differential equations.

C. G. Maple (Ames, Iowa)

2479:

Gibson, R. E. A heat conduction problem involving a specified moving boundary. Quart. Appl. Math. **16** (1958), 426–430.

The standard boundary value problem: (D.E.) $u_t = r^{-k}(r^k u_r)_r + A(t)$, $0\leq r\leq R(t)$, $t>0$, $k=1$ or 2, (B.C.) $u(R(t), t)=u_0$, $u_r(0, t)=0$, is considered in the special cases (1) $R=ct^{1/2}$, $A=at^s$, $s>-1$, and (2) $R=ct$, A being integrable. In the first case, involved manipulations lead to the not unexpected result

$$u = u_0 + a(s+1)^{-1}t^{s+1}[1-F(r/t^{1/2})],$$

where F is a solution of a certain linear, homogeneous second order equation. In the second case the author uses the representation of u involving the Poisson formula with undetermined intensity g. The latter is "determined" on observing that when $R=ct$ the first boundary condition produces essentially the Laplace transform of g. The author fails to realize that this representation is possible only if $A(t)$ is analytic in a suitable t domain.

I. I. Kolodner (Albuquerque, N.M.)

QUANTUM MECHANICS

See also 2392, 2472.

2480:

Shirokov, Iu. M. A group-theoretical consideration of the basis of relativistic quantum mechanics. I. The general properties of the inhomogeneous Lorentz group. Soviet Physics JETP **33** (6) (1958), 664–673 (861–872 Z. Eksper. Teoret. Fiz.).

This is the first of a series of papers on the application of group theory to relativistic quantum mechanics. The program is essentially that of E. Wigner [Ann. of Math. (2) **40** (1939), 149–204], and V. Bargmann and E. Wigner [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 211–223; MR **9**, 553], but the author proposes to obtain the same results by a much simpler method and to extend the considerations to include non-unitary representations and inversions. {In the opinion of the reviewer, the author's treatment as seen in this and the papers reviewed below has two shortcomings. It does not start from the physical problem: what is given in relativistic quantum mechanics is a realization of the inhomogeneous Lorentz group by ray correspondences, i.e., to each Lorentz transformation there corresponds a mapping of the rays of a Hilbert space of states onto themselves which preserves transition probabilities. The problem is to classify such realizations. It was shown by Wigner that this problem can be reduced to that of finding all unitarily inequivalent representations up to a factor of the inhomogeneous Lorentz group by unitary operators (unitary, that is, unless the Lorentz transformation reverses the direction of time, when the operators are anti-unitary). The non-unitary representations discussed in the present paper could only have physical relevance if they appear (a) in a theory with indefinite metric, (b) in the transformations of fields as opposed to states. The second shortcoming is that the author's discussion (which is based on the infinitesimal method) is purely formal, all delicacies arising from the unboundedness of the infinitesimal operators being ignored.} After defining the Lorentz group and the notion of irreducibility, the author discusses the relation between a representation V and the corresponding representations $(V^{-1})^T$, $(V^*)^T$, V^{*-1}, where $*$ is hermitean adjoint and T is transpose. He proves that a necessary and sufficient condition that V and V^{*-1} be equivalent is that the representation leave invariant a nonsingular hermitean form, and that this form is unique up to a factor for an irreducible representation. A representation V is called real by the author if it is equivalent to V^{*-1}; it is called unitary if it is real and there is an operator h which satisfies $hVh^{-1}=V^{*-1}$ and is positive definite. For representations which are complex (i.e., not real), there is no invariant non-singular form, but by taking the direct sum of V and $(V^{-1})^T$ one gets a reducible real representation. This result is exemplified in the case of the spinor representations of the homogeneous Lorentz group. The commutation relations of the infinitesimal operators of the group P_μ (space-time translation) and $M_{\mu\nu}$ (homogeneous Lorentz transformation) with each other and with the inversion operators of space I_s, of time I_t, and space-time I_{st} are displayed. From them the author constructs the operators $g_\mu = M_{\mu\nu}P_\nu$ and

$$\Gamma_\varkappa = (2i)^{-1}\,\varepsilon_{\varkappa\mu\nu\rho}M_{\mu\nu}P_\rho,$$

which satisfy $g_\mu P_\mu = 0$, $\Gamma_\mu P_\mu = 0$. The commutation rules

of these operators with each other and P are given and the invariant operators $P_\mu{}^2$, $\Gamma_\mu{}^2$ are constructed. It is pointed out that $\Gamma_\mu = \Sigma P_\mu$, with Σ a real number, is consistent with the commutation relations, provided $P_\mu{}^2 = 0$, while the operators $M_{\mu\nu}{}^2$ and $\varepsilon_{\mu\nu\kappa\lambda} M_{\mu\nu} M_{\kappa\lambda}$ are invariant only if $P_\mu = 0$. Finally, the following criterion is derived: an irreducible representation is real if and only if the eigenvalues of all the invariants are real for this representation. For real irreducible representations, the eigenvalues of all the components of the operators $M_{\mu\nu}$, P_μ, Γ_σ are also real.

A. S. Wightman (Princeton, N.J.)

2481:

Shirokov, Iu. M. A group-theoretical consideration of the basis of relativistic quantum mechanics. II. Classification of the irreducible representations of the inhomogeneous Lorentz group. Soviet Physics. JETP **33** (6) (1958), 919–928 (1196–1207 Ž. Eksper. Teoret. Fiz.).

This paper continues that reviewed above, in which representations of the Lie algebra of the inhomogeneous Lorentz group were discussed and all invariants of these representations found. Here, a maximal commutative set of elements of these representations is chosen and certain irreducible representations are realized in such a way as to diagonalize these operators. The irreducible representations are classified according to the character of the spectrum of the infinitesimal translation operators whose points are four vectors p. Class P_m: p time-like; class P_0: p light-like; class P_π: p space-like; class O_0: $p = 0$ (the author admits the possibility of non-unitary representations, but considers only those in which the representation of the translation subgroup is unitary). For the class P_m only unitary representations are found, just those obtained by Wigner [Op. cit., review above]. {And to the reviewer it seems that the derivation adds nothing to the treatment of Bargmann and Wigner.} For the class P_π, the author considers the subgroup that leaves the vector $(0, 0, \pi, 0)$ invariant (the so-called little group) and derives commutation relations for the corresponding infinitesimal operators. He determines the form of these infinitesimal operators in terms of the basic invariants. He then finds which of the representations of the Lie algebra so obtained extend to the whole group, which are irreducible, which real. Explicit formulae are given for the infinitesimal operators in each irreducible representation. Since it is known from the above-mentioned work of Wigner that in the unitary case the representation of the group is determined up to equivalence by the representation of the little group, and all the irreducible unitary representations of the little group for the present case were determined by V. Bargmann [ibid. **48** (1947), 568–640; MR **9**, 133], the main new result is the determination of the non-unitary representations.

A. S. Wightman (Princeton, N.J.)

2482:

Shirokov, Iu. M. A group-theoretical consideration of the basis of relativistic quantum mechanics. III. Irreducible representations of the classes P_0 and O_0, and the non-completely-reducible representations of the inhomogeneous Lorentz group. Soviet Physics. JETP **33** (6) (1958), 929–935 (1208–1214 Ž. Eksper. Teoret. Fiz.).

This paper continues those reviewed above. First, irreducible representations of Class P_0 are considered. The discussion follows the method used for P_π. The basic invariants are $P_\mu{}^2 = 0$, and $\Gamma_\sigma{}^2 = \alpha$. For $\alpha = 0$, and $\alpha > 0$, the formulae obtained coincide with those of Wigner, and Bargmann and Wigner. For $\alpha < 0$ and α complex, new representations are listed, but a detailed investigation of their structure is not given. For the Class O_0 it is shown that the representations are essentially those of the homogeneous Lorentz group and a brief summary is given of the results of Gel'fand and Naĭmark [M. A. Naĭmark, Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4 (62), 19–63; Amer. Math. Soc. Transl. (2) **6** (1957), 379–458; MR **16**, 566]. Finally, certain reducible but not completely reducible representations are discussed which occur in the Gel'fand-Yaglom theory of wave equations [Ž. Eksper. Teoret. Fiz. **18** (1948), 703–733; MR **10**, 583].

A. S. Wightman (Princeton, N.J.)

2483:

Shirokov, Iu. M. A group-theoretical consideration of the basis of relativistic quantum mechanics. IV. Space reflections in quantum theory. Soviet Physics. JETP **34** (7) (1958), 493–498 (717–724 Ž. Eksper. Teoret. Fiz.).

The author continues his study of representations of the inhomogeneous Lorentz group, here considering the effect of adjoining space inversion to the connected group. Using infinitesimal methods he reproduces the results of Wigner [op. cit., #2480] and extends them in certain respects. (Wigner only considered those representations which are relevant for classifying the transformation properties of physical systems.) He finds that two irreducible representations of the extended group are equivalent if, and only if, the representations of the extended little group are equivalent. For the representations of the extended little group there are two possibilities: if $g \to U(g)$ is equivalent to $g \to U(I_s g I_s)$ (I_s is space inversion, $g \in$ little group), then the irreducible representation of the extended group contains just one representation of the corrected group; if $g \to U(g)$ and $g \to U(I_s g I_s)$ are inequivalent, the irreducible representation of the extended group contains the two associated inequivalent representations of the connected group. He applies his results in a brief general discussion of the intrinsic parity of elementary particles and recent experiments on parity conservation. On the latter subject, the author takes the following point of view: "In pseudo-Euclidean space, the space inversion and time displacement transformations commute so that it automatically follows that the inversion operator I_s commutes with the Hamiltonian. Consequently, since the experiments of Wu, Lederman and others do not contradict the pseudo-Euclidean character of space-time they cannot contradict the law of conservation of parity, but merely show that the present definition of parity is incorrect."

{In the reviewer's opinion this represents a revision of the basic ideas of relativistic invariance in quantum mechanics which only serves to confuse the situation. It is true that every representation of the restricted group can be extended to the group including space inversion (if zero-mass positive-spin representations are excluded) but the inversion operation so defined will in general have nothing to do with the physical operation of passing from state to space-inverted state, at a particular time. This operation may or may not commute with the operation of translation in time. In the author's opinion the notion of parity should be defined in terms of the operation of

combined inversion, CP, i.e., space inversion followed by charge conjugation. Since, in nature, CP seems to satisfy the correct commutation relations with the infinitesimal operators of the inhomogeneous group the corresponding parity is conserved. However, the same experiments which are consistent with CP's being good, indicate quite unambiguously that P is no good. It could have happened that neither was good; then the author would have had to look for yet another operation in terms of which to define his parity.) *A. S. Wightman* (Princeton, N.J.)

2484:

Ermolaev, A. M. Fock's expansion for the wave functions of many-electron atoms. Vestnik Leningrad. Univ. **13** (1958), no. 22, 48–64. (Russian. English summary)

By means of the theory of harmonic functions on a hypersphere in the Euclidean space of $3N$ Cartesian coordinates of the electrons and the cores, it is shown that a wide class of atomic wave functions may be expanded in the neighbourhood of $r=0$ in a double series, whose terms contain ascending integral powers of both r and $\ln r$. The coefficients of this series are finite, continuous and simple functions of spherical angles in every point on the hypersphere; and they may be found in succession from an infinite system of the Poisson's equations on the hypersphere. There are reasons for supposing that the expansion is convergent for any finite r. The expansion is uniquely determined if the asymptotic expression of the wave functions is given at $r\to\infty$.

P. Roman (Manchester)

2485:

Trifonov, E. D. The correspondence between the irreducible representations of rotation and permutation groups. Vestnik Leningrad. Univ. **13** (1958), no. 22, 157–162. (Russian. English summary)

The author obtains certain correspondences between irreducible representations of rotation groups and irreducible representations of permutation groups. The correspondence is set up in terms of a tensor of rank n in the space of irreducible representations of the rotation group having weight L. The problem arises in connection with certain quantum mechanical systems, consisting of equivalent subsystems, which are simultaneously invariant with respect to rotations of 3-dimensional space and arbitrary permutations of the subsystem.

C. E. Rickart (New Haven, Conn.)

2486:

Mead, C. A. Theory of radiative damping in stationary states. Phys. Rev. (2) **112** (1958), 1843–1847.

As is well known certain interactions transform stationary states of a quantum system into unstable states with a finite lifetime. An example is provided by the electromagnetic interaction between an excited atom and the photon field. Conventional perturbation theory, adapted to the case where the interaction transforms each stationary state into a "perturbed" stationary state, is not suited for the case of unstable states. The present paper gives a formalism by which one can decide whether a given unperturbed stationary state is transformed by an interaction into either a stationary or an unstable state. The case of the Lee model is treated as an example. Another more realistic example, the propagation of a photon in an ideal gas of atoms which can be excited by photon absorption, is described very briefly.

L. Van Hove (Utrecht)

2487:

Chang, T. S. A note on interaction representation. Sci. Record (N.S.) **2** (1958), 250–252.

The Tomonaga-Schwinger equation is derived on the basis of P. Weiss' theory [Proc. Roy. Soc. London Ser. A **156** (1936), 192–220]. *A. J. Coleman* (Toronto, Ont.)

2488:

Prats, Francisco; and Toll, John S. Construction of the Dirac equation central potential from phase shifts and bound states. Phys. Rev. (2) **113** (1959), 363–370.

This paper discusses the extension from the non-relativistic Schrödinger to the relativistic Dirac equation of the work of Jost and Kohn, Moses and Kay on the quantum theory problem of constructing a central potential that will yield a given set of scattering phase shifts and bound-state energy eigenvalues. Intended for physicists, it does not attempt a mathematically complete discussion. Also unfortunately, it depends for many details on the unpublished PhD thesis of D. Carter [Princeton, 1952]. *F. H. Brownell* (Seattle, Wash.)

2489:

Giambiagi, J. J.; and Kibble, T. W. B. Jost functions and dispersion relations. Ann. Physics **7** (1959), 39–51.

In order to investigate the scattering of a Schrödinger particle by a spherically symmetrical potential Jost [Helv. Phys. Acta **20** (1947), 256–266; MR **9**, 555] introduced an auxiliary function $f(k)$, where k is the incident wave number. M. Baker [Ann. Physics **4** (1958), 271–305; MR **19**, 1236] showed that more general cases can be treated by using the expression

$$g(E) = \lim_{\varepsilon\to 0} \mathrm{Det}\,[(E+i\varepsilon-H_0)^{-1}(E+i\varepsilon-H)],$$

where Det means that the determinant has to be taken in an appropriate subspace. In the present paper the properties and applications of $g(E)$ are discussed. It is analytic and bounded in the upper half plane, so that its real and imaginary parts on the real axis are Hilbert transforms. It is suggested that this may be used to 'improve' the Born approximation, although this improvement does not mean that the next order in the potential is obtained correctly. It is shown, however, that in the charged scalar theory the crossing relation does make it possible to find the g^4-term once the g^2-term is known. {Reference 13 should be: Phil. Mag. **42** (1951), 851.}

N. G. van Kampen (Utrecht)

2490:

Drell, S. D.; and Fubini, S. Higher electromagnetic corrections to electron-proton scattering. Phys. Rev. (2) **113** (1959), 741–744.

The validity of the form-factor analysis of electron-proton scattering is discussed. Corrections of order e^4 arise (a) from diagrams related to virtual photon Compton scattering by the proton and (b) from radiative correction to the form-factor analysis. By consideration of (a), the

form-factor analysis is shown to be valid up to approximately 1 Bev for all scattering angles. The methods of dispersion theory are used. *C. Strachan* (Aberdeen)

2491:

Khutsishvili, G. R.; and Cheishvili, O. D. Double elastic scattering of deuterons in a magnetic field. Soviet Physics. JETP **35** (8) (1959), 859–862 (1226–1231 Ž. Eksper. Teoret. Fiz.).

A theoretical treatment is given of the double elastic scattering of a beam of deuterons in a homogeneous magnetic field. It is shown that although experimental measurement of the scattering gives additional information about the scattering amplitude, there is still not sufficient information to determine it completely.

D. F. Mayers (Oxford)

2492:

Eftimiu, C. The forward elastic scattering of photons by bound electrons. Com. Acad. R. P. Romîne **8** (1958), 153–158. (Romanian. English and Russian summaries)

The forward scattering cross-section for the Rayleigh effect has been computed by using the dispersion relation.

Author's summary

2493:

Klarsfeld, S. Sur le schéma Lagrangien dans la théorie des champs. Com. Acad. R. P. Romîne **8** (1958), 159–163. (Romanian. Russian and French summaries)

On montre que, si la fonction de Lagrange

$$L = L(\psi_A, \psi_{A,\alpha})$$

est un invariant quadratique construit avec les potentiels ψ_A et leurs dérivées (ce qui correspond aux théories linéaires), alors la symétrisation du tenseur canonique d'énergie-impulsion ne peut être réalisée par un changement de fonction de Lagrange d'après la formule $L' = L + \partial_\alpha \Lambda_\alpha$. *Résumé de l'auteur*

2494:

Blokhintsev, D. I. On a possible limit on the applicability of quantum electrodynamics. Soviet Physics. JETP **35** (8) (1959), 174–176 (254–257 Ž. Eksper. Teoret. Fiz.).

Identical with Nuovo Cimento (10) **9** (1958), 925–929 [MR **20** #5657].

2495:

Kibble, T. W. B. On the consistency of Schwinger's action principle. Nuovo Cimento (10) **10** (1958), 417–427. (Italian summary)

Schwinger's Action Principle and the resulting field equations and commutation relations are investigated for consistency, using a Lagrangian of considerable generality, with field variables and matrices partitioned to refer to boson and fermion variables. The constancy in time of various commutators on a space-like surface provides a criterion involving a symmetrization of interaction terms which is that required in invariance under "strong reflexion" [Pauli, *Neils Bohr and the development of physics*, pp. 30–51, Pergamon, London, 1955; MR **17**, 692]. *C. Strachan* (Aberdeen)

2496:

Takabayasi, Takehiko. Symétries internes des particules élémentaires et de leurs interactions. Cahiers de Phys. **13** (1959), 27–33.

In this article the author enumerates the transformation properties of the field variables of various field theories under certain transformations in space-time and under certain other operations such as charge conjugation. Auxiliary spaces are introduced in order to interpret these transformations. *A. H. Taub* (Urbana, Ill.)

2497:

Baranger, Michel. General impact theory of pressure broadening. Phys. Rev. (2) **112** (1958), 855–865.

In two earlier papers [Phys. Rev. (2) **111** (1958), 481–493, 494–504; MR **20** #5651, #5652] the author has given a quantum-mechanical formulation of the theory of pressure (collision) broadening of spectral lines. In the present paper the various considerations and approximations of the earlier work are systematized and extended. The fundamental physical condition which is required for the validity of the theory, which is called the impact approximation, is that strong collisions between the radiating atom and surrounding perturbing atoms must be infrequent compared with weak collisions. The major contribution of the quantum-mechanical considerations is the relating of the shifts and widths of the lines to the scattering matrix for collisions of the radiating atom with individual perturbing systems.

E. L. Hill (Minneapolis, Minn.)

2498:

Buimistrov, V. M. On the precision of quantum mechanical mean values computed by the variational method. Soviet Physics. JETP **35** (8) (1959), 812–814 (1161–1164 Ž. Eksper. Teoret. Fiz.).

Kato [J. Phys. Soc. Japan **4** (1949), 334–339; MR **12**, 447] gave a method for estimating the accuracy with which the variational method yields the eigenvalues of a linear self adjoint operator. This paper applies the method to certain calculated quantum mechanical mean values. Unfortunately, the accuracy obtained in the example given of the diamagnetic susceptibility of the helium atom is only twenty per cent. *G. Temple* (Oxford)

2499:

Halpern, Francis R. Meson kinematics in static source meson theories. Ann. Physics **7** (1959), 146–153.

This paper aims to give a convenient set of many-meson states in static source meson theory. The states are diagonal in angular momentum and isotopic spin, and are otherwise defined to form bases for the irreducible representations of the permutation group. The simpler examples are written out explicitly.

J. C. Taylor (London)

2500:

Halpern, F. R.; Sartori, L.; Nishimura, K.; and Spitzer, R. The physical nucleon in static source meson theory. Ann. Physics **7** (1959), 154–173.

This paper aims to give a reliable approximation to the static source, cut-off meson theory model of the physical nucleon; and hence to judge the model. The approxima-

tion consists in expanding the physical nucleon state vector as a polynomial in the Hamiltonian, and determining the coefficients by a minimization procedure. It is taken to a stage at which 5 virtual mesons are involved, where the convergence appears to be rapid.

Reasons are adduced why the static model should not be expected to be reliable for quantities which depend on the nucleon operators. The use of sum rules which are, within the model, exact confirms this conclusion. For quantities which depend on the meson operators, the method gives similar results to the Chew-Low one-meson approximation, in spite of the fact that many-meson states (in the sense of the paper under review) are not found to be negligible. Thus the vector part of the anomalous magnetic moment comes out right, but the electron-neutron interaction does not. *J. C. Taylor* (London)

2501:

Lurié, David. Une généralisation de la transformation de Foldy-Wouthuysen. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 577–588. (English summary)

The author writes the equations given by Whittaker [Proc. Roy. Soc. London Ser. A **121** (1928), 543–557] and Bhabba [Rev. Mod. Phys. **21** (1949), 451–462] for a particle of spin 1 in Hamiltonian form using three anti-commuting 12×12 matrices. It is then shown that two of these matrices may be eliminated by a transformation analogous to the Foldy-Wouthuysen transformation [L. Foldy and S. A. Wouthuysen, Phys. Rev. (2) **78** (1950), 29–36]. *A. H. Taub* (Urbana, Ill.)

2502:

Fiutak, J. Dirac and Gürsey equations for bosons. Nuovo Cimento (10) **10** (1958), 292–299. (Italian summary)

The author uses two index four component and two index two component spinors to write the wave equations for bosons. The most of the results of this paper are known [cf. A. H. Taub Phys. Rev. (2) **56** (1939), 799–810; **57** (1940), 807–814; MR **1** 95, 352].

A. H. Taub (Urbana, Ill.)

2503:

★The many body problem. Le problème à *N* corps. Université de Grenoble: Cours donnés à l'école d'été de physique théorique, Les Houches, 1958. Methuen, London; John Wiley and Sons, Inc., New York; Dunod, Paris; 1959. xv+675 pp. $15.00.

Within the last few years there has been a considerable development in both the conceptual aspects and mathematical techniques relating to the investigation of the behaviour of many particles in interaction with one another—the so-called many body problem. In particular, a much greater understanding of the relation between individual particle motion and collective motion has been achieved and also the way in which particle motions are correlated.

In 1958 at the Summer School in Theoretical Physics at Les Houches a course dealing with most aspects of the many body problem was given by a number of experts in this particular field. The book under review is a reproduction of the lectures and seminars given during the course.

The first article, by N. M. Hugenholtz, sets out a formulation of perturbation theory for many body systems in which only two body interactions play a part and in which perturbation theory is valid. This latter requirement imposes severe restrictions on the interaction and means that in many cases of physical interest the theory cannot be used.

The next contribution occupies one quarter of the volume and is an account by K. A. Brueckner of the work he initiated on the many body treatment of nuclear matter. He shows how to obtain a rapidly convergent series for the nuclear energy although that for the wave function diverges. Using an accurate two-body interaction derived from experiment he is then able to predict correctly the properties of nuclear matter. He goes on to apply his methods to the study of liquid He^3 and is able to give an excellent semi-quantitative description of the system. Further applications are made to the properties of an electron gas in the presence of a uniform background of positive charge and a system of interacting bosons.

D. J. Thouless discusses a new approach to the many-body problem which to some extent clears up some of the difficulties and ambiguities of Brueckner's work.

C. Bloch and C. de Dominicis obtain an approximate expression for the Grand Partition Function for a gas of Fermions undergoing binary collisions.

B. R. Mottelson gives a survey of the physical ideas underlying the description of nuclear spectra. He considers the independent particle model, quadrupole polarization effects, two particle spectra and the effects of short and long range interactions between nucleons. The nuclear spectra of deformed nuclei are then discussed with particular accent on the existence of an energy gap in even-even nuclei.

A short qualitative article by V. F. Weisskopf gives an explanation of the fact that although strong forces act between nucleons, they nevertheless move as approximately independent particles within the nuclear volume.

H. J. Lipkin gives a short discussion of the relation between independent and collective particle motion in particularly simple systems. He further examines the problems resulting from the introduction of spurious degrees of freedom for the description of collective motion.

S. T. Beliaev gives a full account of the Bogoliubov canonical transformation method in which the low-lying states of a many body system are described in terms of a number of elementary excitations or quasi-particles. The method is then applied to a system composed of a large number of fermions interacting through a "pairing" force and a superconducting state is obtained which is separated from the lowest excited state by an energy gap Δ. In view of the similarity of this gap to that occurring in nuclei (cf. the earlier article by Mottelson) an application of the method is also made to nuclei in which it is again assumed that a strong pairing correlation exists. A number of semi-quantitative explanations of nuclear properties follow in a straightforward fashion.

D. Bohm in a long article presents the general theory of collective coordinates. He discusses elementary plasma theory, a treatment dealing with the individual particle character in terms of a field and the method of subsidiary conditions in which both field and particles are introduced but related through a subsidiary condition. A quantum mechanical treatment of the methods described is also given.

An account of the general features of a free electron gas and a gas of electrons and ions interacting via the Coulomb field is given by D. Pines. The major topics covered are the random phase approximation and a discussion of its validity together with the dielectric formulation of the many body problem and its application to the free electron gas.

J. R. Schrieffer gives an excellent account of the theory of superconductivity. In particular the successful theory of Bardeen, Cooper and Schrieffer which is able to give both a qualitative and quantitative understanding of many important features of superconductivity is presented in some detail.

Three seminars by E. A. Lynton describe the most recent experiments dealing with the energy gap and the nature of the relevant electromagnetic equations in the superconducting state.

Finally, K. Huang describes the method of pseudo potentials as applied to a hard-sphere Bose gas and obtains the ground and excited state energies by Rayleigh-Schrödinger perturbation theory and a new perturbation method. In a second article he describes a method developed by T. D. Lee and C. N. Yang for the calculation of the partition function of a many body system whose particles interact with each other through a general two-body potential.

In all, this collection of articles represents an exceedingly full account of most aspects of the many body problem and will be an invaluable reference book for anyone about to engage or already engaged in this field of study. *R. J. Blin-Stoyle* (Cambridge, Mass.)

RELATIVITY

See also 2289, 2290, 2291, 2292, 2293, 2294, 2464, 2465.

2504:

Hillion, Pierre. Limite à la vitesse de la lumière d'un système particulier de paramètres d'Einstein-Kramers. C. R. Acad. Sci. Paris **248** (1959), 2731–2733.

The author studies the behavior of a particular orthogonal ennuple of vectors associated with a particle having a four-velocity vector u^μ and an intrinsic angular momentum described by the anti-symmetic tensor $s^{\mu\nu}$ in the limit as the vector u^μ is made to approach a null vector. *A. H. Taub* (Urbana, Ill.)

2505:

Hillion, Pierre. Limite à la vitesse de la lumière d'un système particulier de paramètres d'Einstein-Kramers. C. R. Acad. Sci. Paris **248** (1959), 2853–2855.

This is a continuation of a previous note [reviewed above] in which the limit of two parameters occuring therein are evaluated. *A. H. Taub* (Urbana, Ill.)

2506:

Pignedoli, Antonio. Sul moto di una particella di energia relativistica nel campo magnetico di una spira percorsa da corrente elettrica costante, cui si sovrapponga un campo elettrico uniforme, diretto secondo l'asse della spira. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 256–281. (English summary)

The motion of a charged particle in a magnetic field produced by a plane coil and a superposed uniform axial field is studied using the classical relativistic equations of motion. In particular, motion along the axis, in the plane of the coil (for $E=0$ only), and near the axis are considered in detail. (From the author's summary) *C. A. Hurst* (Adelaide)

2507:

Takabayasi, Takehiko. L'espace de parité du champ électromagnétique. C. R. Acad. Sci. Paris **248** (1959), 70–73.

The author points out the known fact that the Maxwell free field is invariant under the transformation

$$F_{\mu\nu} \to F_{\mu\nu}' = aF_{\mu\nu} + b\hat{F}_{\mu\nu},$$

where $\hat{F}_{\mu\nu}$ is the pseudo-tensor dual to the antisymmetric tensor $F_{\mu\nu}$ and a and b are constants satisfying

$$a^2 + b^2 = 1.$$

[cf. Misner and Wheeler, Ann. Physics **2** (1957), 525–603; MR **19**, 1237.] *A. H. Taub* (Urbana, Ill.)

2508a:

Lenoir, Marcel. Principe d'une théorie unitaire. Interprétation basée sur l'emploi d'un espace fibré. C. R. Acad. Sci. Paris **248** (1959), 1944–1946.

2508b:

Lenoir, Marcel. Principe d'une théorie unitaire. Interprétation basée sur l'emploi d'une hypersurface d'un espace pentadimensionnel. C. R. Acad. Sci. Paris **248** (1959), 2074–2075.

Fibre spaces are used to set up a structure on V_n with an affine connexion and three 1-forms, covariant, contravariant and scalar. With $n=4$ and a fundamental tensor given, field equations are obtained from a variational principle, and when the various additional tensors are specialised in different ways, the equations of Einstein-Schrödinger, Tonnelat and Bonnor are obtained.

In the second paper the curvature forms used in the first paper are derived from the study of V_4 as a sub-space of a V_5. *A. G. Walker* (Liverpool)

2509:

Lichnerowicz, André. Ondes et radiations électromagnétiques et gravitationnelles. C. R. Acad. Sci. Paris **248** (1959), 2728–2730.

A singular 2-form is a non-zero six-vector $\phi_{\alpha\beta}$ such that there exists a vector l_α (necessarily null) and

$$l^\alpha \phi_{\alpha\beta} = 0, \qquad l_{(\alpha}\phi_{\beta\gamma)} = 0.$$

Similarly for a singular double 2-form $H_{\alpha\beta\lambda\mu}$ (which has the symmetry of the curvature tensor). Then

$$H_{\alpha\beta\lambda\mu} = \sum_{i,j=1}^{2} a_{ij}(l_\alpha n_\beta^{(i)} - l_\beta n_\alpha^{(i)})(l_\lambda n_\mu^{(j)} - l_\mu n_\lambda^{(j)}),$$

where $n_\alpha^{(1)}$, $n_\alpha^{(2)}$ are two unit orthogonal vectors tangent to the light cone on which l_α lies.

In the Einstein-Maxwell electrodynamics consider a surface S of discontinuity. The jump in the curvature

tensor defines a singular double 2-form, whose l_α is normal to S, and that in $F_{\alpha\beta;\gamma}$ is $l_\gamma\phi_{\alpha\beta}$, where $\phi_{\alpha\beta}$ is a singular 2-form. Pure radiation is now defined as a state in which the curvature tensor itself defines a singular double 2-form. For such a state the invariants (trace and discriminant) of the quadratic form satisfy equations of continuity along l_α. *C. W. Kilmister* (London)

2510:

★**Séminaire de mécanique analytique et de mécanique céleste, dirigé par Maurice Janet; 1re année: 1957/58.** Secrétariat mathématique, Paris, 1958. iii+104 pp. (mimeographed)

Table des matières: Y. Fourès-Bruhat, La relativité générale; F. Hennequin, Équations approchées du mouvement en relativité générale (methode du tenseur impulsion-énergie); J. Lévy, Corrections de relativité dans le mouvement des planètes; Pham Tan Hoang, Étude des équations du mouvement en relativité générale par la méthode des singularités; A. Kolmogorov, Théorie générale des systèmes dynamiques de la mécanique classique; S. Kichenassamy, Choix des solutions particulières à symétrie axiale en relativité générale; C. Lanczos, Remarks concerning the canonical formulation of field equations. L'article de Mme Fourès-Bruhat présente un bref sommaire des principes mathématiques de la relativité générale et du problème de mouvement. L'article de Kolmogorov est une traduction du texte russe de la conférence faite par l'auteur au Congrès International des Mathématiciens à Amsterdam en 1954 [vol. 1, pp. 315–333, Noordhoff, Groningen, 1957; MR **20** #4066].

A. Trautman (Varsovie)

2511:

Beresford Rayner, Charles. Sur une solution générale des équations intérieures d'Einstein pour un mouvement de groupe. C. R. Acad. Sci. Paris **248** (1959), 2725–2727.

The author obtains an explicit solution of the interior form of Einstein's field equations which also satisfy the Killing equations. He deduces a simplified form of the Einstein equations representing conditions of rigidity.

M. Wyman (Edmonton, Alta.)

2512:

Narlikar, V. V. A brief survey of some problems in gravitation. Bull. Calcutta Math. Soc. **49** (1957), 1–7.

2513:

Narlikar, V. V. A brief survey of some problems in gravitation. II. Bull. Calcutta Math. Soc. **50** (1958), 1–5.

2514:

Moffat, John. The foundations of a generalization of gravitation theory. Proc. Cambridge Philos. Soc. **53** (1957), 473–488.

The author examines a unified field theory in which the metric tensor is symmetric but complex. The imaginary (symmetric) part of the metric tensor is to represent the electromagnetic potentials, which thus have a structure similar to that of Maxwell theory only in the "slow" approximation. There is no investigation of the spin character of the "electromagnetic" waves in this theory, but from the information provided in the paper it is clear that this spin is 2, a result that throws grave doubts on the physical viability of the theory proposed.

P. G. Bergmann (New York, N.Y.)

2515:

Takeno, Hyôitirô. On plane wave solutions of field equations in general relativity. Tensor (N.S.) **7** (1957), 97–102.

The author determines new solutions of the Einstein field equations for a space-time with a metric of the form

$$ds^2 = -A(dx^2+dy^2)-B(dz^2-dt^2),$$

where A and B are functions of $Z=x-t$. The energy tensor is taken to be that of a Maxwell field $F_{\mu\nu}$ with no charges or currents present such that

$$F_{12} = F_{34} = 0$$

and the remaining components of $F_{\mu\nu}$ are functions of Z alone. The author does not make use of the simplification $B\equiv 1$ which can be made by a coordinate transformation. The solutions given in this paper are similar to those previously obtained by the author [Tensor (N.S.) **6** (1956), 15–25; MR **18**, 704]. *A. H. Taub* (Urbana, Ill.)

2516a:

Takeno, Hyôitirô. On some generalized plane wave solutions of non-symmetric unified field theories. I, II. Tensor (N.S.) **7** (1957), 34–58; **8** (1958), 71–78.

2516b:

Takeno, Hyôitirô. On plane wave solutions of field equations in general relativity. II. Tensor (N.S.) **8** (1958), 59–70.

The first part of the first paper is divided into two sections. In the first the author investigates plane electromagnetic waves in Einstein's (Schrödinger's) unified field theory. The symmetric part $h_{\lambda\mu}=g_{(\lambda\mu)}$ of the fundamental tensor $g_{\lambda\mu}$ is spherically symmetric and conformally flat. Under this assumption he finds an approximate solution. {Reviewer's remark: Methods for obtaining approximate solutions for more general cases are given in Hlavatý, J. Rational Mech. Anal. **2** (1953), 1–52 [MR **14**, 505]. In the approximation of Schrödinger case the space-time must be of constant curvature. Cf. Hlavatý, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 1052–1058 [MR **14**, 505].} In the second section the author considers Taub's space-time with plane symmetry and finds an approximate as well as an exact solution in the Einstein case and an approximate solution in the Schrödinger case. All these solutions belong to the third class. [For an exact solution of the second class see Hlavatý, J. Rational Mech. Anal. **3** (1954), 103–146; MR **15**, 654; for a brief discussion of the author's results from the point of view of the reviewer's classification and research, see Tensor (N.S.) **7** (1957), 141–142; MR **19**, 1239.] In the second part of this paper the author applies previous methods to the case where the electromagnetic tensor field $F_{\lambda\mu}$ is of the form

$$F_{\lambda\mu} = (\sqrt{-g})^{-1}\omega_{\lambda\mu\nu\sigma}g^{\nu\sigma}.$$

The symmetric part of $g_{\lambda\mu}$ leads to the metric

$$(1)\qquad ds^2 = -Adx^2-2Ddxdy-Bdy^2 -(C-E)dz^2-2Edzdt+(C+E)dt^2$$

and $A, \cdots, E$ are functions of $Z=z-t$. Using the reviewer's method for $\Gamma^{\nu}_{[\lambda\mu]}$ [*Geometry of Einstein's unified field theory*, Noordhoff, Groningen, 1957; MR **20** #5067], he obtains exact solutions of the third class with vanishing charge current density vector.

In the second paper the author investigates the Maxwell Einstein theory

$$K_{\lambda\mu} = -8\pi E_{\lambda\mu} \tag{2}$$

of a purely radiational electromagnetic field $F_{\lambda\mu}$ taking the metric (1). In the absence of the electromagnetic field ($K_{\lambda\mu}=0$) the functions A, B, C, D are related by one condition, while E is arbitrary. In the remaining case the solution of (2) is given by (1) and

$$F_{14} = F_{31} = \sigma, \quad F_{42} = F_{32} = \rho, \quad F_{12} = F_{34},$$

where σ and ρ are related to A, B, C, D by one equation. In the appendix the author finds the unique parallel vector field v^{ν} of (1). {Reviewer's remark: The holonomy group of (1) has two parameters. It yields by algebraic means the field v^{ν} and its uniqueness as well as all other metrics with the Christoffel symbols of (1).}

V. Hlavatý (Bloomington, Ind.)

2517:

Rao, J. R. Rigorous solution of Einstein's unified theory for a special type of symmetry. Acta Phys. Austriaca **12** (1958/59), 251–261.

Let $g_{ij}(x^1, x^2, x^3, x^4)$ be the field tensor of the weak form of Einstein's unified field theory. The author assumes that, for $x_1=\text{const}$, $x_4=\text{const}$, g_{ij} is homogeneous and isotropic in x_2 and x_3. Although the author is not able to find the general solution of the field equations under these conditions, he is able to obtain a particular solution. The paper discusses some of the physical properties of this particular solution. *M. Wyman* (Edmonton, Alta.)

2518:

Tonnelat, Marie-Antoinette. Représentation de la matière en relativité générale et en théorie unitaire. Cahiers de Phys. **13** (1959), 1–11.

Exposé d'ensemble de l'état actuel du problème de l'unification des champ, du lieu entre le champ et ses sources (en particulier dans la théorie de Born et Infeld et la théorie d'Einstein et Schrödinger), ainsi que des résultats obtenus, et des problèmes restant ouverts, pour la quantification du champ. *Y. Fourès-Bruhat* (Reims)

2519:

Bel, Louis. Quelques remarques sur la classification de Petrov. Étude du cas 2. C. R. Acad. Sci. Paris **248** (1959), 2561–2563.

L'A. signale, dans les cas 2 et 3 de la classification de Petrov du tenseur de courbure [Kazan Gos. Univ. Uč. Zap. **114** (1954), no. 8, 55–69; MR **17**, 892], l'existence d'un vecteur isotrope l tel que

$$R_{\alpha\beta,\lambda\mu}l^{\alpha}l^{\lambda} = Kl_{\beta}l_{\mu}, \quad R^{*}_{\alpha\beta,\lambda\mu}l^{\alpha}l^{\lambda} = K^{*}l_{\beta}l_{\mu}.$$

Le tenseur T défini par l'A. [C. R. Acad. Sci. Paris **248** (1959), 1207–1300; MR **21** #1194] vérifie

$$T_{\alpha\beta,\lambda\mu}l^{\beta}l^{\lambda}l^{\mu} = 0.$$

L'A. montre que dans le cas 2 les trajectoires de l sont des géodésiques isotropes. *Y. Fourès-Bruhat* (Reims)

2520:

Schmutzer, Ernst. Beitrag zur projektiven Relativitätstheorie. II. Z. Physik **154** (1959), 312–318.

In this paper the author discusses briefly the physical significance of various quantities occurring in an earlier paper [Z. Physik **149** (1957), 329–339; MR **19**, 927] on Jordan's generalization of projective relativity. In addition the field equations satisfied by a scalar field are derived from a variational principle. Further the field equations are specialized to the case of a spherically symmetric point mass and to the case where the line element is that of a cosmological space. For the latter case no exact solutions are known.

A. H. Taub (Urbana, Ill.)

2521:

Gião, Antonio. Field equations of any differentiable variety. Portugal. Math. **17** (1958), 63–83.

If $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the first and second groundforms of a V_n imbedded in a Riemannian V_{n+1}, and $G_{\alpha\beta}$ and $H_{\alpha\beta}$ are the Einstein tensors formed from $a_{\alpha\beta}$ and $b_{\alpha\beta}$ respectively, then the author considers the "field equations", $G_{\alpha\beta}-\gamma a_{\alpha\beta}=T_{\alpha\beta}$, and $H_{\alpha\beta}-\delta b_{\alpha\beta}=U_{\alpha\beta}$, where $T_{\alpha\beta}$ and $U_{\alpha\beta}$ are symmetric tensors. By counting conditions, including the Gauss and Codazzi equations, the author concludes that a necessary condition that $a_{\alpha\beta}$, $b_{\alpha\beta}$, $T_{\alpha\beta}$ and $U_{\alpha\beta}$ be determined by the above equations, is that $n=4$. Further, by imposing the requirement that the solution be determined by a "natural" set of Cauchy data, it follows that V_n must be of class 1 with a normal hyperbolic metric. These results convince the author that his equations are more significant than the field equations of general Relativity. The values of γ and δ are studied, the existence of a family of space-like subspaces of V_4 with elliptic metric is established, and a term is obtained which "has important consequences in astrophysics (formation of the spiral arms of the galaxies)".

A. J. Coleman (Toronto, Ont.)

ASTRONOMY

See also 2461, 2510.

2522:

Gurzadian, G. A. The dynamics of planetary nebulae. Voprosy Kosmog. **6** (1958), 157–210. (1 plate) (Russian. English summary)

The problem of the dynamics of planetary nebulae has two aspects. The first concerns the origin of the planetary nebulae. The second—the forces, acting upon the gaseous envelope surrounding a hot star, and the possible courses of the evolution of its form and structure under the action of these forces. The main types of planetary nebulae can be assumed as: (1) stellar, (2) planet-like, (3) two-envelope, (4) ring-formed, (5) bipolar, (6) spiral, (7) diffuse (or transitional). It is shown in § 2 that the two-envelope nebulae cannot be formed as a result of repeated outbursts of the central nucleus. They originate at a definite stage of evolution of the planetary nebulae and the two-envelope phase is an inevitable consequence of their evolution. In § 3 the existence of planet-like (i.e. having only one envelope) planetary nebulae is proved. The evolution of planetary nebulae proceeds along the follow-

ing sequence: planet-like→two-envelope→ring-formed. In § 4 the ring nebulae are examined. It is shown that: (1) the density of matter in the central parts of the ring nebulae NGC 6720 and NGC 7293 is about one-third of that of the envelope itself. (2) The electron temperature in the central parts of the ring nebulae is considerably (3–5 times) larger than its generally assumed value for planetary nebulae (10,000° K) which is the result of insufficient cooling caused by ions of oxygen, nitrogen, sulphur, etc. This means that the planetary nebulae are not isothermic gaseous objects in their whole volume. § 5: Proceeding from the study of the structure of planetary bipolar nebulae, an attempt is made to show the existence of magnetic fields in the planetary nebulae. Two possibilities are examined: (a) the nucleus has a bipolar field, which redistributes the ionized matter of the nebulae; (b) the magnetic field can be carried away by the matter released somehow from the central star. The magnetic fields of the galaxy do not play any rôle in the dynamics of planetary nebulae. § 6: Apparently, the spiral planetary nebulae are in close relation with the bipolar ones. §§ 7, 8: The spatial structure of the planetary nebulae is mainly spherical. Any different details in their structure such as snails, arcs, spirals, etc., are secondary. These details suggest nonstable processes in the nuclei of planetary nebulae. § 9: A summary is given of the quantitative theory of the origin of the second envelope as a result of tearing-off by the pressure of L_α radiation. The tear-off takes place during the stage when the optical thickness of the planetary nebula in ultraviolet frequences is equal to a few units. § 11: The initial velocities of the outbursts of gaseous matter from the nucleus cannot be of the order of 1000 km per sec, as no known mechanism can explain the deceleration of their expansion to the order of 10 km per sec. § 12: Examination of some problems connected with the hydrodynamics of the planetary nebular is made. § 13 is devoted to the problem of the origin of planetary nebulae in connection with the origin and evolution of stars in general. There are no known stars in our Galaxy from which planetary nebulae could be formed. The planetary nebulae are formed as sub-products from the excessive mass, which is not used up during star formation. In the case of dwarfs such a picture is observed in the Herbig-Haro objects.

The evolution of the nuclei of planetary nebulae is going on along with the evolution of the planetary nebulae, and the moment the nebula disappears the nucleus becomes an ordinary star of the Galaxy. A considerable part of stars of the intermediate population of the Galaxy are former nuclei of planetary nebulae.

Author's summary

2523:

Safronov, V. S. On the growth of terrestrial planets. Voprosy Kosmog. **6** (1958), 62–77. (Russian. English summary)

The process of growth of the embryos of planets resulting from the fall of small particles and of larger bodies on them is considered. In absence of crushing of bodies, small particles were swept out by the larger bodies at the early stage of planetary growth. The cloud transformed into a "swarm" and the further process of growth consisted principally in the fall of individual bodies on the embryos. The crushing of bodies at their collisions prevented the swarm from a complete sweeping out of small particles. A considerable part of planetary matter was exposed to repeated crushings and cohesions. Gaseous matter took almost no part in the process.

The rate of the earth's growth is considered, supposing the relative velocities of bodies to be determined by gravitational interaction with the largest bodies and the planet embryo. The heating of the earth due to the shocks caused by the falling bodies, to the radioactive decay and to the compression of the growing planet is estimated. The growth of the earth had actually finished by the end of the first quarter of billion years. By that time the temperature in the earth's centre attained 1500° K. The surface of the earth always remained cold.

It is pointed out in conclusion that the process of growth of major planets was essentially different.

Author's summary

2524:

Agekian, T. A. The velocity distribution function and the rate of dissipation in systems of gravitating bodies. Astr. Ž. **36** (1959), 283–294. (Russian. English summary)

The author examines the dissipation in systems in the case where the field stars have the same mass as that of the star under consideration and in the case where the field stars all have the same mass but the considered star has zero mass. His calculations, based on his theory of probability of approach presented in an earlier paper [same Ž. **35** (1958)], show that the rate of dissipation in clusters is higher by a factor of 30 to 40 in the latter case than in the former case. *R. G. Langebartel* (Urbana, Ill.)

2525:

Ščigolev, B. M. Approximate calculation of ephemerides in the restricted three-body problem. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 37–48. (Russian)

Consider the motion of an asteroid (of mass zero) under the attraction of the Sun (of mass one) and the planet Jupiter, the motion of which relative to the Sun is determined by tables calculated from the analytic theory of motion of this planet. Such a scheme is closer to the reality than the restricted elliptical one because in the tabulated values of the coordinates of Jupiter the perturbations due to the presence of the other planets are taken into account.

The paper describes an averaging method which differs from those usually considered in celestial mechanics by the fact that not the force function is averaged over the whole period of time but certain other terms which are selected in such a way that the resulting differential equations of motion are integrable in finite terms. In the paper is elaborated the averaging of the cube of the inverse distance between the perturbed and the perturbing bodies, this term causing the main difficulty in the integration of the differential equations. Finally the integration of the averaged equations is discussed.

E. Leimanis (Vancouver, B.C.)

2526:

Veiga de Oliveira, F. Solutions homographiques du problème généralisé des n corps. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 5–82. (Portuguese. French summary)

The author considers a generalized n-body problem in which the bodies are acted on by internal forces derived from a potential homogeneous of degree $\alpha+1$, $\alpha \neq -3$. Homographic solutions are characterized and shown to exist. The stability of certain rigid homographic solutions is investigated. Many of the results are generalizations of those given by Wintner [*The analytical foundations of celestial mechanics*, Princeton Univ. Press, Princeton, N.J., 1941; **3**, 215; pp. 294–306].

W. Kaplan (Ann Arbor, Mich.)

2527:

Krook, Max. Structure of stellar atmospheres. II. Astrophys. J. **129** (1959), 724–733.

The related paper I was published with the title "On the solutions of equations of transfer" [same J. **122** (1955), 488–497; MR **17**, 543]. The author's abstract of the present paper reads: "The structure equations for non-gray atmospheres can generally be solved only by means of iteration procedures. A moment method is presented for solving the equation of transfer within the framework of such an iteration scheme. Certain average coefficients are evaluated from solutions of monochromatic-moment equations in which the argument T of the Planck function is identified with a known 'trial' temperature. Those coefficients are then inserted in a set of formal integrated-moment equations in which T is now identified with the required 'improved' value of the temperature distribution."

2528:

Brown, Archibald. On the stability of a hydromagnetic prominence model. Astr. J. **128** (1958), 646–663.

A treatment of two dimensional solutions of the equilibrium equations of fluid magnetics

$$\nabla p + \rho g \mathbf{k} = \frac{\mu}{4\pi} \operatorname{curl} \mathbf{H} \times \mathbf{H},$$

$$\operatorname{div} \mathbf{H} = 0.$$

The solutions of this system are given in terms of an arbitrary function $M(F)$ where F satisfies

$$\nabla^2 F = M(F) \exp\left(-\frac{z}{h_0}\right),$$

and p and $\mathbf{H}$ are given by

$$p = -\frac{\mu}{4\pi} \exp\left(-\frac{z}{h_0}\right) \int M(F) dF, \quad \mathbf{H} = \left(0, -\frac{\partial F}{\partial z}, \frac{\partial F}{\partial y}\right);$$

here h_0 is the scale height, $h_0 = p/\rho g$. The stability question for certain equilibrium solutions against two-dimensional perturbations is discussed in terms of a variational principle.

A. A. Blank (New York, N.Y.)

GEOPHYSICS

See 2461.

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 2207.

2529:

★Pfanzagl, J. Die axiomatischen Grundlagen einer allgemeinen Theorie des Messens. Schriftenreihe des Statistischen Instituts der Universität Wien. N. F. Nr. 1. Physica-Verlag, Würzburg, 1959. 63 pp. DM 14.00.

M sei eine Menge, der eine Struktur aufgeprägt sei. Das Problem des Messens besteht darin, M isomorph bezüglich der gegebenen Struktur in die Menge der reellen Zahlen abzubilden. Zunächst wird der Fall betrachtet, dass die Struktur von M durch eine Vollordnung V_1 ihrer Elemente gegeben ist und M in der durch V_1 induzierten Topologie zusammenhängend ist. Verwandte Ergebnisse bei G. Debreu [*Decision processes*, edited by R. M. Thrall, C. H. Coombs and R. L. Davis, Wiley, New York; Chapman and Hall; London; 1954; MR **16**, 605] und bei G. Cantor [Math. Ann. **46** (1895), 481–512], worauf Verf. hinweist. Der Hauptteil der Arbeit ist aber dem Fall gewidmet, dass neben V_1 auch eine mit V_1 verträgliche Vollordnung V_2 der Paare von Elementen aus M gegeben ist. {Bem. d. Ref.: Die Wahl desselben Symbols für die verschiedenen Ordnungsrelationen V_1 und V_2 scheint weiterhin keine Unstimmigkeiten zur Folge zu haben.} Erweitert man V_2 durch ein Stetigkeits- und Vertauschbarkeits-Axiom zu einem Axiomensystem A, dann erweist sich A als äquivalent einem Axiomensystem D der Mittelbildung für die Elemente von M. D ist eine Formalisierung eines von J. Aczél [Bull. Amer. Math. Soc. **54** (1948), 392–400; MR **9**, 501] angegebenen Systems. Jedge Menge M, deren Struktur durch V_1 und A (oder D) gegeben ist, lässt sich isomorph auf die Menge der reellen Zahlen abbilden, wobei der Mittelbildung die Addition entspricht. Diese Abbildung ist bis auf lineare Transformationen der reellen Geraden in sich eindeutig. Dieser Satz steht mit bekannten Spezialfällen in Relation (Abstandsdefinitionen, Einführung eines projektiven Koordinatensystems auf einer Geraden). Der Verf. studiert insbesondere die Anwendung auf die v. Neumann-Morgensternsche Messung des subjektiven Nutzens. Vgl. hiezu auch Verf.: A general theory of measurement-applications to utility, Princeton Univ., Econometric Research Program, Res. Mem. No. 5, 1958.

L. Schmetterer (Berkeley, Calif.)

2530:

Arrow, Kenneth J.; and Nerlove, Marc. A note on expectations and stability. Econometrica **26** (1958), 297–305.

The subject of this article is the problem of stability of a dynamized Walrasian system where the excess demand functions depend on expected prices as well as on current prices. The problem is studied under the assumption that expected prices depend in a simple way on the rate of change of prices.

T. Haavelmo (Oslo)

2531:

Arrow, Kenneth J.; and McManus, Maurice. A note on dynamic stability. Econometrica **26** (1958), 448–454.

The article deals with conditions of stability for systems of differential equations of a type frequently occurring in

economic theory. Some new theorems concerning the matrices of coefficients characterizing such systems are given. *T. Haavelmo* (Oslo)

2532:

★**Markowitz, Harry M. Portfolio selection: Efficient diversification of investments.** Cowles Foundation for Research in Economics at Yale University, Monograph 16. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. x+344 pp. $7.50.

In this monograph the general ideas of the theory of programming and the algebra of maximizing functions subject to constraints in the form of inequalities have been applied to the practical problem of selecting "good" portfolios among a large number of securities. This problem can be described briefly as follows. Suppose that a portfolio can be characterized by a certain number of attributes (as, e.g., expected yield, variability of yield from year to year, etc.). Assume that there is a unique direction of preference attached to each of these attributes. Finally, assume that enough information is available regarding each security to make it possible to calculate the values of the portfolio attributes for every conceivable portfolio. The problem is, first, to organize the possible portfolios so as to facilitate choice between them, and second, if possible, to develop rational procedures for selecting a "best" portfolio. The subjective data required from a client are the list of attributes which he wants to take into account, and a principle of "weighing" the importance of one attribute against others.

It is possible to read the book without too much knowledge of mathematical technique. The book itself develops the mathematics it needs. However, there is no denying that the mathematically trained reader will have a much better chance of understanding the book than the non-mathematician. *T. Haavelmo* (Oslo)

2533:

★**Sasieni, Maurice; Yaspan, Arthur; and Friedman, Lawrence. Operations research: Methods and problems.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xi+316 pp. $10.25.

This book covers a number of the newly emerging fields of operations research, inventory, queuing, game theory, and programming, by working out simple examples which tend to illustrate both the idiosyncrasies of actual operations and the principles of the mathematical tools for analysis of the operational situation. General theories are not discussed; specific examples, approximating to actual operational situations, are described and outlines of the methods of solution of the problem are given, in numerical detail. This is an apotheosis of the case history method. It should be useful for persons not familiar with abstract methods, who wish to find out something about operations research. *P. M. Morse* (Cambridge, Mass.)

2534:

Urech, Aug. Quelques aspects des capitaux différés et des rentes sur plusieurs têtes. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 75–97. (German, Italian and English summaries)

The author uses the well-known Z-formulae [see for instance A. Berger, *Mathematik der Lebensversicherung*, Springer, Wien, 1939, § 42] to compute symmetric pure endowment insurances (annuities) for joint lives.

E. Lukacs (Washington, D.C.)

2535:

Bellman, Richard; and Dreyfus, Stuart. A bottleneck situation involving interdependent industries. Naval Res. Logist. Quart. **5** (1958), 307–314.

This paper considers the topical problem of how to maximize production with two interdependent industries, the motor car and the steel industries being taken as an example. A mathematical model is built up and under the assumption of proportional costs and returns the computational problem of finding a numerical solution is discussed in detail. The linear programming approach is considered first and then the problem is restated in terms of Bellman's dynamic programming approach.

This simplifies the computing problem considerably by reducing the maximum number of steps required to find a solution. For example, a thirty step process, including ninety variables subject to one hundred and twenty constraints, reduces to the determination of a sequence of one-dimensional functions over a region of transformations of shrinking size, that is, a problem of T^2 steps is reduced to one of $2T$ steps.

The computing technique is discussed in detail and a good flow diagram is given for those wishing to code up similar problems on electronic computors.

L. J. Slater (Cambridge, England)

2536:

Courtillot, Marcel. Programmation linéaire. Étude de la modification de tous les paramètres. Méthode de résolution séquentielle. C. R. Acad. Sci. Paris **247** (1958), 670–673.

The essential content of this note has been known and applied for some six or seven years in the United States and elsewhere, and particularly important parts thereof have even been coded for digital machines under the name of "parametric programming." The terminology and basic theorems appear to be those of Charnes, Cooper and Henderson, *Introduction to linear programming* [Wiley, New York, 1953; MR **15**, 48]. The paper contains no references. *A. Charnes* (Evanston, Ill.)

2537:

Simpson, Kenneth F., Jr. In-process inventories. Operations Res. **6** (1958), 863–873.

The problem considered is that of determining the number of intermediate inventories and their levels within a large industrial concern. The concern operates on a "base-stock" system where final items are provided from stock on demand and replacement orders down the line are given at that time. It is assumed that a level of maximum reasonable demand to be met from inventory at each stage has been set by company policy; that demand at each inventory is a random uncorrelated variable; and that inventory costs are proportional to average inventory. These assumptions lead to a cost function

$$C = c + \sum_{i=1}^{i=n} r_i \sqrt{(S_{i-1} - S_i + T_i)},$$

where S_i is service time at stage i, T_i is processing time (fixed), and c and r_i are constants. Total cost is to be minimized with respect to the S_i. In the nature of the problem $0 \leq S_i \leq S_{i-1} + T_i$, ($S_0 = 0$, $S_n = 0$) and therefore the minimum of a concave function under linear constraints is required. This minimum will occur at an extreme point of a convex polyhedron as defined by the constraints, and is thus easy to calculate. If average inventory at any stage is found to be zero, tight "coupling" will result in the elimination of that point as an inventory point.

G. Morton (London)

2538:

★**Shubik, Martin. Edgeworth market games.** Contributions to the theory of games, Vol. IV, pp. 267–278. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

An n-person trading game involving two commodities is considered. The set I of n players is partitioned into a set M of players initially having equal amounts of commodity 1 and none of commodity 2 and a set N of players initially having equal amounts of commodity 2 and none of commodity 1. Assuming that the utility preference functions of all n players are identical the characteristic function of the game is exhibited in a parametrizable form and the following types of results are obtained and investigated: (1) a solution for the case $|M| = |N|$ (where $|A|$ = the number of elements in A); (2) the behavior of the core for $|M| = 1$ and $|N| \to \infty$; (3) the behavior of the core for $n \to \infty$ and $|M|/|N| = k$. All of these results are given economic interpretations and the von Neumann-Morgenstern theory for these examples is compared to that of Edgeworth. *H. Raiffa* (Cambridge, Mass.)

2539:

★**Griesmer, James H. Extreme games with three values.** Contributions to the theory of games, Vol. IV, pp. 189–212. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

An assignment of flows to the non-empty coalitions of an n-person game is a real-valued function I such that (i) $I(I_n) = 0$ if I_n is the whole player set, (ii) $I(\{i\}) = 0$ if $i \in I_n$, and (iii) $I(S) + I(T) = I(S \cup T)$ if $v(S) + v(T) = v(S \cup T)$ and $S \cap T = \emptyset$, where v is the characteristic function of the game. Theorem: An n-person game is extreme if and only if the only flow assignment is $I(S) = 0$. The coalition systems of constant sum essential games with three values are characterized. A linear graph associated with intermediate coalitions is used to decide extremeness questions. Properties of the system of inequalities, which define the set of all (0, 1) normalized n-person constant sum games, are given which characterize extreme games with two values and extreme games with three values are described: in some cases a minimal winning coalition can completely discriminate against the remaining players; in other cases two disjoint minimal intermediate coalitions discriminate against the complement of their union.

S. Sherman (Philadelphia, Pa.)

2540:

Sakaguchi, Minoru. On a certain multi-stage game. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **6** (1959), 1–4.

Bellman and Kalaba [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 749–751; MR **19**, 824] showed how information theory could be interpreted in terms of certain multi-stage decisions processes arising in statistical communication theory. The present paper generalizes the model considered there by considering nature as an opponent with the probabilities of the random events at each stage. Using the functional equation technique of dynamic programing, the author obtains the optimal strategies for each player, and the value of the game. As he indicates, this represents an extension also of the "candy problem" of Steinhaus. *R. Bellman* (Santa Monica, Calif.)

2541:

Sakaguchi, Minoru. Values of strategic information. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **6** (1959), 5–12.

The author considers some games in which one side has partial information concerning the play of the other side. This concept is formulated in precise terms, and some min-max theorems are established, using linear inequality theory. *R. Bellman* (Santa Monica, Calif.)

2542:

Hoggatt, Austin C. An experimental business game. Behavioral Sci. **4** (1959), 192–203.

2543:

Gale, David; and Gross, Oliver. A note on polynomial and separable games. Pacific J. Math. **8** (1958), 735–741.

A two-person zero-sum game is separable if its payoff function is of the form $M(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y)$, where x and y are elements of strategy sets X and Y. The game is a polynomial game if X and Y are bounded subsets of Euclidean spaces and $M(x, y)$ is a polynomial in the coordinates of x and y. The authors prove the following theorems. If X and Y are bounded subsets of Euclidean spaces whose closures contain infinitely many cluster points, then for any finite mixed strategies μ and ν there exists a polynomial payoff function M such that the associated game has μ and ν as its unique optimal strategies. If X and Y are metric spaces containing infinitely many points and μ and ν are any finite mixed strategies on X and Y respectively, then there is a payoff M, bounded continuous and separable on $X \times Y$, such that the associated game has μ and ν as unique optimal strategies. *E. D. Nering* (Boulder, Colo.)

BIOLOGY AND SOCIOLOGY

See also 2317.

2544:

Barakat, Richard. A note on the transient stage of the random dispersal of logistic populations. Bull. Math. Biophys. **21** (1959), 141–151.

"The transient stage of the random dispersal of logistic populations is investigated, using a Sturm-Liouville

series leading to an infinite system of non-linear integral equations. These equations are then solved via a successive approximation scheme. R. A. Fisher's (steady-state) velocity of advance paradox is discussed. An illustrative example is worked to the second order of approximation." (Author's summary) *J. L. Snell* (Hanover, N.H.)

2545:

Landahl, H. D. A note on population growth under random dispersal. Bull. Math. Biophys. **21** (1959), 153–159.

"An approximation method using a sine function is used to solve the second degree growth equation for the case in which an organism may simultaneously become dispersed throughout a uniform region. The resulting expression for a special case is compared with the expression obtained by R. Barakat (see above review) giving the first two terms, by an iterative procedure. The agreement is satisfactory." (Author's summary)

J. L. Snell (Hanover, N.H.)

2546:

Kempthorne, Oscar; and Nordskog, Arne W. Restricted selection indices. Biometrics **15** (1959), 10–19.

H. F. Smith [Ann. Eugenics **7** (1936), 240–250] developed an index for the selection of plant lines, using R. A. Fisher's concept of discriminant function to derive a linear equation based on observable characteristics as the best available guide to the "genetic value" of each line. L. N. Hazel [Genetics **28** (1943), 476–490] extended this technique to cover the selection of individuals whose progeny shall be of superior "breeding value". The authors add the restriction that certain components of the value shall not change, and derive a corresponding "restricted selection index". The demand for this arises in poultry breeding, and some numerical examples in this context are given. *I. M. H. Etherington* (Edinburgh)

2547:

Greenhouse, Samuel W.; and Geisser, Seymour. On methods in the analysis of profile data. Psychometrika **24** (1959), 95–112.

"This paper is concerned with methods for analyzing quantitative, non-categorical profile data, e.g., a battery of tests given to individuals in one or more groups. It is assumed that the variables have a multinormal distribution with an arbitrary variance-covariance matrix. Approximate procedures based on classical analysis of variance are presented, including an adjustment to the degrees of freedom resulting in conservative F tests. These can be applied to the case where the variance-covariance matrices differ from group to group. In addition, exact generalized multivariate analysis methods are discussed. Examples are given illustrating both techniques." (From the authors' summary)

M. Muller (New York, N.Y.)

INFORMATION AND COMMUNICATION THEORY

See also 2035a-b, 2036a-b, 2037, 2325, 2540, 2541.

2548:

Matveev, P. S. A method of determining optimal impulsive response for a class of inputs. Avtomat. i Telemeh. **20** (1959), 3–15. (Russian. English summary)

The author extends the work of Zadeh and Ragazzini [J. Appl. Phys. **21** (1950), 645–655; MR **12**, 347] to the case where $g(t)$, the expectation of the message component, admits of the representation

$$g(t) = \sum_{\nu=0}^{n} a_\nu P_\nu(t), \quad P_\nu(t-\tau) \equiv \sum_{\mu=1}^{r} b_\mu{}^\nu(\tau)\psi_\mu{}^\nu(t),$$

in which the a_ν are unknown constants and the $b_\mu{}^\nu$ and $\psi_\mu{}^\nu$ are linearly independent functions. Such cases have been considered previously by several authors, notably Dolph and Woodbury [Trans. Amer. Math. Soc. **72** (1952), 519–550; MR **14**, 295] and M. Blum [Trans. I.R.E. **IT-2** (1956), 76–81], but the solution given in the present paper is more explicit. *L. A. Zadeh* (Berkeley, Calif.)

2549:

Hatori, Hirohisa. A note on the entropy of a continuous distribution. Kōdai Math. Sem. Rep. **10** (1958), 172–176.

The author derives from a certain set of postulates the formula

$$H = -\lambda \int f(x_1 \cdots x_n) \log f(x_1 \cdots x_n) dx_1 \cdots dx_n$$

for the entropy of a probability distribution with density $f(x_1 \cdots x_n)$ in E_n. {The reviewer wishes to point out that the fairly plausible additional postulate that entropy is not increased by data grouping requires $H = +\infty$ for any probability distribution which is not purely atomic [see I. B. Fleischer, Bull. Amer. Math. Soc. **63** (1957) 400].}

S. P. Lloyd (Murray Hill, N.J.)

2550:

Kramer, H. P. A generalized sampling theorem. J. Math. Phys. **38** (1959/60), 68–72.

The author generalizes the "sampling" representation

$$f(t) = \lim_{N\to\infty} \sum_{|n|<N} t(n/2w)(2\pi wt - n\pi)^{-1} \sin(2\pi wt - n\pi),$$

valid for $f(t) \in L^2(-\infty, \infty)$, $f(t) = \int_{-W}^{W} e^{2\pi i x t} g(x)dx$, to a form $f(t) = \lim_{N\to\infty} \sum_{|n|<N} f(t_k) S_k(t)$. Here the t_k are eigenvalues of a particular nth order differential operator, and the S_k are functions derived from the eigen-functions. The theorem includes both the classical cardinal series and sampling at the zeros of $J_n(\sqrt{z})$; the details are too involved to reproduce here.

H. O. Pollak (Murray Hill, N.J.)

2551:

Dobrušin, R. L. A statistical problem in the theory of detection of signals in the background of noise in a multichannel system, reducing to stable distribution laws. Teor. Veroyatnost. i Primenen. **3** (1958), 173–185. (Russian. English summary)

Under certain simplifying assumptions, the problem of detecting the presence of a signal in one of n Rayleigh channels can be reduced to the problem of discriminating between the densities

$$p_A = \prod_{i=1}^{n} \frac{2x_i}{d} \exp\left(-\frac{x_i^2}{d}\right),$$

$$p_B = \frac{1}{n} \sum_{j=1}^{n} \frac{2x_j}{d+\bar{d}} \exp\left(-x_j^2/(d+\bar{d})\right) \prod_{i\neq j} \frac{2x_i}{d} \exp\left(-\frac{x_i^2}{d}\right),$$

where $d=E(\xi_i^2)$ in the absence of signal, $d+\bar{d}=E(\xi_i^2)$ in the presence of signal, ξ_i being the output of ith channel. Assume that the false alarm probability p_1 is not greater than F and the detection probability p_2 is not smaller than D, where F and D satisfy $0<F\leq D<1$. Let $\beta_n(F, D)$ be a value of the ratio $\bar{d}/d$ such that for $\bar{d}/d\geq\beta_n(F, D)$ the inequalities $p_1\leq F$ and $p_2\geq D$ cannot be realized with any likelihood ratio test, while for $\bar{d}/d<\beta_n(F, D)$ there exists a likelihood ratio test for which the inequalities are satisfied. Using some results of Gnedenko and Kolmogoroff [*Predel'nye raspredeleniya dlya summ nezavisimyh slučaĭnyh veličin*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, Moscow, 1949; MR **12**, 839] and V. M. Zolotarev concerning stable distributions, the author obtains several asymptotic forms for $\beta_n(F, D)$, one of which reads

$$\beta_n(F, D)=\frac{\log n+\log 1/F}{\log 1/D}-1+O(1)$$

for fixed D and n as $F\to 0$.

L. A. Zadeh (Berkeley, Calif.)

2552:

Deman, Pierre. Spectre instantané et analyse du signal simultanément en fréquences et en temps. Application aux télécommunications. Ann. Télécommun. **14** (1959), 21–32.

1re Partie.—En essayant de généraliser ce que l'on appelle une transmission de signaux on fait apparaître les conditions dans lesquelles une transmission d'information est possible: une part d'inconnu dans un contexte connu.—On s'efforce d'analyser les paramètres dont dépend le contexte connu et la suite des opérations par lesquelles on peut parvenir à saisir les inconnues dans le signal reçu.—Cette analyse conduit à introduire une notion de spectre instantané d'amplitude qui permet de montrer que la localisation de l'énergie simultanément en fréquence et en temps n'admet pas une solution unique.—On termine sur quelques conséquences théoriques et pratiques montrant la fécondité de la méthode qui élimine quelques paradoxes des méthodes classiques d'analyse.

2e Partie.—On montre sur des exemples simples la transposition des opérateurs mathématiques dans la technique habituelle. Tout en ouvrant la voie vers des dispositifs originaux, on montre comment on peut utiliser plus complètement les renseignements recueillis avec les procédés classiques de filtrage.—On donne une méthode génétale de construction d'une voie de transmission à haut rendement informationnel ainsi que les moyens à mettre en oeuvre pour éliminer au mieux les erreurs de transmission dues aux signaux parasites d'interférence avec d'autres voies de transmission ou à un bruit incohérent. *Résumé de l'auteur*

2553:

du Castel, François; et Derennes, Jeanne. Réception en diversité de deux signaux corrélés. Ann. Télécommun. **14** (1959), 41–42.

Les auteurs étudient l'influence d'une certaine corrélation entre deux signaux sur le gain de diversité. Ils étendent au cas des signaux ayant une distribution gaussienne un calcul effectué par H. Staras dans le cas d'une distribution de Rayleigh. *Résumé des auteurs*

SERVOMECHANISMS AND CONTROL

See also 1935, 2473.

2554:

★**Bushaw, D. Optimal discontinuous forcing terms.** Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 29–52. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. $3.75.

The results of this paper were obtained by the author in 1952 in his Princeton dissertation and were circulated in report form in 1953. The problem arises in the design of a feedback control system, and it was the author's work that brought this problem to the attention of mathematicians. The author restricts himself to a system with one degree of freedom: $\ddot{x}+g(x, \dot{x})=\varphi(x, \dot{x})$ where the control function φ is restricted to the values -1 and $+1$. The problem is to select the control φ to bring the system to its equilibrium state in minimum time. The problem is solved in the linear case $g(x, y)=Ax+By$. The method of solving the problem is ingenious but does not generalize. For more recent work and further references, see a paper by Bellman, Glicksberg, and Gross [Q. Appl. Math. **14** (1956), 11–18; MR **17**, 1206] and one by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 573–577].

J. P. LaSalle (Baltimore, Md.)

2555:

Merriam, C. W., III. A class of optimum control systems. J. Franklin Inst. **267** (1959), 267–281.

Consider a dynamical system characterized by a relation of the form

$$q^{(n)}(t)=F[q^{(n-1)}(t), \cdots, q(t);\ m^{(r)}(t), \cdots, m(t)],$$

where m is the control variable, q is the response variable, and $r\leq n$. Assume a criterion function of the form

$$\epsilon=\int_t^{t+\tau}\{\lambda(\sigma)f_q[Q(\sigma)-q(\sigma)]+f_m[M(\sigma)-m(\sigma)]\}d\sigma,$$

where Q and M are estimates of q and m, respectively, over $[t, t+\tau]$, f_q and f_m are strictly concave and λ is nonnegative. The problem is to determine m over $[t, t+\tau]$ which minimizes ϵ. On applying dynamic programming in a purely formal manner, the author obtains a general condition for the minimum which in the special case of a linear system and quadratic criterion function reduces the problem to the solution of a system of ordinary differential equations. An important simplifying feature which was observed by Bellman and others is that under the above assumptions and in the absence of constraints on the control variable the minimal error is a quadratic function of the state variables.

L. A. Zadeh (Berkeley, Calif.)

2556:

Kalman, R. E.; and Bertram, J. E. A unified approach to the theory of sampling systems. J. Franklin Inst. **267** (1959), 405–436.

A sampled-data system is essentially a collection of elements whose behavior is governed by difference or difference-differential equations. The authors develop a unified method of analyzing various types of linear sampled-data systems in which the intervals between successive samples of time-variables are not necessarily of

constant length and/or the sampling instants are not the same for all time-variables. For such systems they construct relations expressing the state at time t_{n+1} in terms of the state at time t_n, where t_{n+1} and t_n represent the beginning and end of a transition during which the behavior of the system is governed by a set of linear differential or difference equations with constant coefficients. The authors' technique provides a very effective tool for the analysis and synthesis of linear sampled-data systems.

L. A. Zadeh (Berkeley, Calif.)

2557:

Bellman, Richard; and Kalaba, Robert. A mathematical theory of adaptive control processes. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1288–1290.

The authors make use of some of the results of dynamic programming to establish a model for adaptive control processes (e.g., man-machine systems). In particular, the theory of decision processes and sufficient statistics is used to construct the model. The paper has 22 references in its bibliography. In the middle of page 1288 read "abstraction" for "abstruction."

A. A. Mullin (Urbana, Ill.)

2558:

Roginsky, V. N. The synthesis of mixed relay circuits of series-parallel type. Avtomat. i Telemeh. **18** (1957), 1120–1131. (Russian. English summary)

A sketch of an algebraic method for dealing with finite winding resistances in multi-contact and multiple-winding relay circuits.

L. A. Zadeh (Berkeley, Calif.)

2559:

Rabin, M. O.; and Scott, D. Finite automata and their decision problems. IBM J. Res. Develop. **3** (1959), 114–125.

According to Rabin and Scott, an automaton is a Turing machine that cannot write on its tapes (it may have more than one). They consider the classification of finite tapes (or pairs, etc., of tapes) obtained by starting an automaton at the beginning of each tape and noting the state of the automaton when the end of one of the tapes is reached. They first consider automata having one tape which can only move in one direction on it and give several characterizations of the sets of tapes which can be defined by such automata. Next they show that two-way one tape automata define only the same sets as the one-way automata. Finally, they show that several questions about the sets of pairs of tapes defined by two tape automata are undecidable.

J. McCarthy (Cambridge, Mass.)

2560:

Shepherdson, J. C. The reduction of two-way automata to one-way automata. IBM J. Res. Develop. **3** (1959), 198–200.

The author gives a simpler proof of Rabin's result [see preceding review] that two-way finite automata, which are allowed to move in both directions along their input tape, are equivalent to one-way automata as far as the classification of input tapes is concerned. The one-way automaton constructed to be equivalent to a given two-way automaton has sufficiently many more states so that the result is not immediately relevant to programs using tape.

J. McCarthy (Cambridge, Mass.)

HISTORY AND BIOGRAPHY

See also 2355.

2561:

★Struik, Dirk J. A matematika rövid története. [A concise history of mathematics.] Gondolat Kiadó, 1958. 219 pp. (10 plates) 30.00 Ft.

A Hungarian translation by Auer Kálmán of the book [Dover, New York, 1948] reviewed in MR **10**, 173.

2562:

Vogel, Kurt. Ist die babylonische Mathematik sumerisch oder akkadisch? Math. Nachr. **18** (1958), 377–382.

The author adduces two reasons for thinking that "Babylonian" mathematics was originated by the Sumerians rather than by the later Akkadians: (1) Most of the Akkadian mathematical technical terminology either was taken over directly from Sumerian or translated literally from it. (2) At an early stage in the development of the cuneiform writing, geometric figures as well as ideograms underwent a rotation through a counter clockwise right angle. Thus a trapezoid, say, with a base properly labelled as "upper width" in the more primitive style of writing, would appear lying on its side in the later style taken over by the Akkadians, but with the left-hand side still retaining the designation of "upper width". Had the Akkadians themselves originated this class of problem they presumably would have designated as "upper" those parts actually upper on the tablets as written by them.

E. S. Kennedy (Beirut)

2563:

van der Waerden, B. L. Babylonische Planetenrechnung. Vierteljschr. Naturf. Ges. Zürich **102** (1957), 39–60.

This is a study of Babylonian methods of calculating planetary phenomena. A brief introduction gives information on some aspects of the Babylonian system. The uninitiated reader can consult O. Neugebauer [*The exact sciences in antiquity*, 2nd ed., Brown Univ. Press, Providence, R.I., 1957; MR **19**, 825] for more details. Another publication by Neugebauer [*Astronomical cuneiform texts*, vols I, II, III, Humphries, London, 1955; MR **17**, 931] contains texts cited in this paper. Two types of texts are involved, ephemerides and procedure texts.

The author first considers two procedure texts which give the velocity schemes of Jupiter and Saturn. In an ephemeris for Jupiter, however, the planet positions are in agreement with the procedure text just previously discussed, but the dates differ. The author concludes that the dates in the procedure text are an approximation and, to correct these, he introduces a rule, called the Sonnenabstandprinzip, which he discusses.

A discussion of the motion of Mars is next undertaken and, after dividing the ecliptic into 133 so-called steps, a rule is deduced for certain positions of Mars. A comparison of the results from this rule are in good agreement with data of a certain procedure text. The time intervals and the elongation of Mars are also discussed in some detail. A consideration of the velocity scheme of Mars shows a possible connection between the Babylonian theory and that of an Egyptian table for Mars.

The last planet considered is Venus for which three

different systems are known. A discussion of these systems is given. Finally, two rather remarkable Venus tables are discussed at some length. Appended is a tabulation of dates during which each of the three Venus systems was in use. *E. B. Allen* (Troy, N.Y.)

2564:

Frenkian, Aram. Recherches de mathématiques suméro-akkadiennes, égyptiennes et grecques. II. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 281–294.

The author continues his sketch of ancient mathematics [part I: same Bull. **1 (49)** (1957), 17–32; MR **20** #2244] by outlining Greek and Egyptian numeral systems and computational methods. In discussing the numerology and number theory of the Pythagoreans he rejects the notion that the latter were inspired by the Babylonians. Since the scientists of Alexandria are classified in this paper as non-Greek, it is possible to maintain the assertion that the Greeks never utilized the principle of place-value. *E. S. Kennedy* (Beirut)

2565:

Szabó, Árpád. Deiknumi, als mathematischer Terminus für "beweisen". Maia (N.S.) **10** (1958), 106–131.

This is an attempt to trace phases in the connotation of the Greek word appearing in the title. As used in the earliest geometry it conveyed its primitive meaning "to show, display, indicate". This is correlated with the concomitant notion of proof as being a process of making the theorem visually evident by means of a figure. Later, in the Pythagorean number theory, the figure no longer played an essential role in proof, and the same word took on the meaning "to make known by words", thence "to prove". Several theorems are given as examples. *E. S. Kennedy* (Beirut)

2566:

Szabó, Árpád. Die Grundlagen in der frühgriechischen Mathematik. Studi Ital. Filologia Classica **30** (1958), 1–51.

The author investigates the question of how it first came to be realized that a deductive mathematical system requires an axiomatic foundation. He remarks that in pre-euclidean geometry there was no standard terminology to denote the "definitions", "postulates", and "common notions" (axioms) of Euclid's Elements, and that there remain embedded in the Elements vestiges of the earlier inconsistencies. At least three levels of development are detectable: (1) the empirical geometry of the Ionians, (2) Pythagorean number theory, and (3) the geometrisation of algebra forced by the discovery of irrationals. In connection with (2) the author argues that emphasis on the indivisibility of the number one was due to influence of the Eleatic school of philosophers, and that the adoption of the axiom "the whole is greater than the part", was in response to the arguments of Zeno, of the same school. This axiom was part of an attempt to put geometry on as rigorous a basis as number theory. Hence it is inferred that the need for axiomatic foundations was realized concurrently with the first efforts to develop self-consistent abstract systems. *E. S. Kennedy* (Beirut)

2567:

Schäffer, J. J. The scientific personality of Archimedes. Fac. Ingen. Agrimens. Montevideo. Publ. Didact. Inst. Mat. Estadist. **1** (1958), 57–93. (Spanish)

The paper begins with a short biographical sketch of Archimedes, followed by a listing and brief discussion of his works. The author characterizes him as the pure type of original investigator, in contrast to Euclid, whom he regards as a master of didactic exposition. Most of the rest of the paper consists of an attempt to retrace the steps by which Archimedes arrived at and proved a body of theory. These include the first known use of heuristic devices to intuit the result, supplemented by rigorous proof of its validity. *E. S. Kennedy* (Beirut)

2568:

Thomas, Ivo. A 12th century paradox of the infinite. J. Symb. Logic **23** (1958), 133–134.

"Adam of Balsham . . . in 1132 A.D. reaches the notion of a set A equinumerous with a set B containing A as a proper subset."

2569:

Alter, George. Two renaissance astronomers: David Gans, Joseph Delmedigo. Rozpravy Československé Akad. Věd. **68** (1958), no. 11, 77 pp. (15 plates)

David Gans (1541–1613) and Joseph Delmedigo (1591–1655) were two scientists who lived in the times of Tycho Brahe, Kepler, and Galileo, when the foundations of modern science were being laid, and somewhat before the publication of Newton's great work, the Principia.

Gans was born in Westphalia, Germany, and later moved to Prague. He was a friend of Tycho Brahe and acquainted with Kepler. He devoted his life to scientific study and writing. A list of his works is given but this paper is concerned with an astronomical textbook [*Nechmad ve-Naim*, Jessnitz, 1743]. A manuscript of this is located in the National Archive of Moravia at Brno. The twelve sections of the book as well as an interesting epilogue, are discussed at considerable length. Gans died at Prague and was buried in the graveyard of the former Ghetto there.

Joseph Delmedigo was born on the island of Candia. During his life he travelled through such countries as Italy, Egypt, Turkey, Poland, Russia, Germany, and Bohemia. He studied at Padua under Galileo who taught him mathematics and astronomy. Also he studied medicine which he practiced for many years. He did much writing but only those texts which were printed have survived. This paper is concerned with one [*Elim*, Amsterdam, 1629] containing his scientific papers, sections of which deal with mathematics and astronomy as well as other scientific matters. He, too, was buried in the graveyard of the former Ghetto at Prague.

Although the lives of Gans and Delmedigo overlapped, there is a great difference in their scientific outlook. The work of Gans, although he knew the work of Copernicus and was acquainted with Tycho Brahe and Kepler, is still the science of the Middle Ages. Delmedigo's work represents the seventeenth century break with medieval science.

Among the plates at the end of the book are photographs of tombstones of Gans and Delmedigo as well as a

portrait of the latter. The book also contains a bibliography of works bearing on the subjects of this article.
E. B. Allen (Troy, N.Y.)

2570:
Rychlík, Karel. Cauchys Schrift "Mémoire sur la dispersion de la lumière" herausgegeben während seines Aufenthaltes in Prag durch die Königliche böhmische Gesellschaft der Wissenschaften. Czechoslovak Math. J. 8 (83) (1958), 619–632. (Russian summary)

2571:
Terracini, Alessandro. Postilla su "Cauchy a Torino". Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 81–82.
Addendum to the paper in same Rend. **16** (1956/57), 159–203 [MR **20** #806].

MISCELLANEOUS

See also 2316, 2317.

2572:
★Band, William. Introduction to mathematical physics. University Physics Series. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-New York-London, 1959. xi+326 pp. $7.25.
The chapters of this undergraduate textbook are titled: I. Introduction: The language of mathematical physics; II. The continuum theory of matter; III. The molecular theory of matter; IV. The theory of fields; V. The theory of relativity; VI. Quantum theory. The treatment of this broad spectrum of subject matter is necessarily introductory in a book of this length (315 pages of text). There are more than 400 exercises which seem interesting and not of the "cookbook" variety. Each chapter closes with a list of references to other textbooks or treatises.
R. A. Toupin (Washington, D.C.)

2573:
★Kronig, R. (Editor) Textbook of physics. 2nd revised English ed. Pergamon Press, New York-London-Paris-Los Angeles, 1959. xiv+961 pp. (2 inserts) $15.00.
This is the second English edition of the Dutch effort to discuss all of physics (at approximately the advanced undergraduate level) in some thousand pages. The results are somewhat irregular, but, on the whole, surprisingly successful. The irregularity is largely a result of the varied authorship of the different chapters, and it is, indeed, surprising that the book is as homogeneous as it is. There is a high level of sensitivity to the fact that physics is the study of the laws of nature, a point that is often obscured in general textbooks. In particular, the sections on instrumentation are excellent, as are the chapters on optics and thermodynamics.
The biographical notes add a great deal of spice to the book, and are often rather blunt. For example: "Debye is a strong and level-headed man; his native country has seen fit to retain him as a professor for only two years." Overall, the book has a great deal to recommend it, though there is no single course for which it is likely to be useful as a text. *H. W. Lewis* (Madison, Wis.)

2574:
Klamkin, M. S. A moving boundary filtration problem or "the cigarette problem". Amer. Math. Monthly **64** (1957), 710–715.
A study is made of filtration through a burning cigarette, assuming that a constant fraction of any component Z in the tobacco is burnt at the tip, the rest being transmitted and filtered down the cigarette, and that the absorption coefficient of the tobacco is constant with respect to the filtration of component Z. Differential equations are set up and solved in the cases of steady and intermittent inhalation. Comparisons are made between cigarettes of different lengths with respect to the amount of Z transmitted per unit length smoked when (B_1) both cigarettes burn down the same amount, or (B_2) both burn down to the same final length. For steady inhalation and (B_2), the filtering capacity turns out to be independent of initial length. In the other cases, the longer cigarette is the better filter. *N. J. Fine* (Princeton, N.J.)

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The Australian Journal of Statistics (Vol. 1, no. 1, dated April 1959), published by the Statistical Society of New South Wales, will appear three times a year, in April, August and November, ten shillings per issue, free to members of the society. Information may be obtained from D. W. Maitland, Bureau of Census and Statistics, Box 796, G.P.O., Sydney, N.S.W.

Cahiers du Centre de mathématique et de statistique appliquées aux sciences sociales. Université Libre de Bruxelles, Institut de Sociologie Solvay, Brussels, Belgium. Cahier no. 1 is dated 1959, with price 150 Belgian francs. "Le Centre compte, à intervalles plus ou moins réguliers, publier des fascicules contenant des articles généraux et des articles originaux consacrés à des méthodes fréquemment utilisées dans ... des problèmes d'ordre économique et des problèmes relatifs à la gestion des entreprises publiques ou privées."

International Journal of Abstracts, Statistical Theory and Method. The first issue is dated July 1959. Published for the International Statistical Institute by Oliver and Boyd Ltd., Tweeddale Court, Edinburgh 1. Annual subscription £5 or $16.00. Single copies 30s. or $4.50. The general editor is Dr. Wm. R. Buckland, c/o 55 Broadway, London, S.W.1, England. The aim of the journal is to give complete coverage of papers in the field of statistical theory, including associated aspects of probability and other mathematical methods and of new contributions to statistical method as published after October 1, 1958. The abstracts (about 400–500 words in length) will be in the English language. The journal will be quarterly and will contain approximately 1000 abstracts per year.

Journal of Mathematical Physics. A bimonthly devoted to new mathematical methods for the solution of physical problems as well as original research in physics furthered by such methods. The first issue is scheduled to appear early in 1960. Subscription rates are: $10 U.S. and Canada; $11 elsewhere. Orders and inquiries should be addressed to the publisher: American Institute of Physics, 335 East 45 St., New York 17, N.Y.

Mathematics of Computation. In January 1960 the title of "Mathematical Tables and Other Aids to Computation" will be changed to "Mathematics of Computation". The general format and the emphasis on mathematical tables will remain the same but there will be increased coverage of modern advances in the theory and application of computational methods. Published quarterly by the National Academy of Sciences, National Research Council, 2101 Constitution Ave., Washington 25, D.C.

Optics and Spectroscopy. A translation of the Russian journal Optika i Spektroskopiya, by Morris D. Friedman, Inc., on the initiative of The Optical Society of America, Inc., with a grant-in-aid from The National Science Foundation. The first issue of the translated publication is dated January 1959, Vol. VI, no. 1, corresponding to the Russian Tom VI, Vyp. 1. Published semi-monthly. Subscriptions for one year to members of the Society: Associates $13.00; Regular members and Fellows $17.00. Subscriptions for non-members: One Year U.S. and Possessions, $25.00; Foreign $28.00. Back numbers: $3.00 per copy. Business and Editorial office: 33 University Road, Cambridge 38, Mass.

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